On the governing fields for tame kernels of quadratic fields

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Aim of this talk

- What is the tame kernel of number fields?
- What is governing fields?
- Why do governing fields matter?
Plan

1. Tame kernel of number fields

2. The governing fields for 2-power ranks of ideal class groups of quadratic fields

3. Known facts about the governing fields for 2-power ranks of ideal class groups of quadratic fields (our model case)

4. some known results for 2-power ranks of tame kernels associated with quadratic fields

5. Hurrelbrink-Kolster’s 4-rank formulae [HK98]

6. toward a governing field for 4-rank of tame kernels associated with quadratic fields
Milnor’s $K_2$ of a number field

$F$: a number field of finite degree over the rationals $\mathbb{Q}$, the second Milnor K-group $K_2(F)$ is defined by

$$K_2(F) := F^\times \otimes F^\times / \langle x \otimes (1 - x) | x \in F^\times, x(1 - x) \neq 0 \rangle.$$  

The class represented by $a \otimes b \in F^\times \otimes F^\times$ is denoted by $\{a, b\} \in K_2(F)$.
Milnor’s $K_2$ of a number field (cont’d)

$S$: a finite set of finite places of $F$, $O_S(F)$: the ring of $S$-integers of $F$, $O_S^\times(F)$: the group of $S$-units of $F$,

$$K_2^S(F) := \{\{a, b\} \in K_2(F) | a, b \in O_S^\times(F)\}.$$  

Note that $K_2^S(F)$ is finitely generated (since $O_S^\times$ is so).

$S_m$: the first $m$ finite places of $F$ with respect to the norm $N(v)$ of $v$,

then it holds that

$$K_2(F) = \lim_{m \to \infty} K_2^{S_m}(F).$$
Tame symbol at a finite place $v$

Let $v$ be a finite place of $F$, $k(v)$ be the residue field at $v$, then the **tame symbol** $\partial_v$ at $v$ is defined by

$$\partial_v : K_2(F) \to k(v)^\times, \quad \{a, b\} \mapsto (-1)^{\alpha\beta} \frac{a^\beta}{b^\alpha} \pmod{v},$$

where $\alpha = \text{ord}_v(a)$, $\beta = \text{ord}_v(b)$, $\text{ord}_v(\cdot)$ is the additive normalized valuation at $v$. 
Tame kernel of number fields

We define the tame kernel $K_2(O_F)$ of a number field $F$ (whose ring of integers $O_F$) to be

$$K_2(O_F) := \bigcap_{v: \text{ fin. places}} \ker(\partial_v : K_2(F) \to k(v)^\times).$$

**Fact.** The tame kernel of number field $F$ is coincide with the second algebraic $K$-group of $O_F$. 
Finiteness of tame kernels

**Fact** (Garland [Gar71]). \( \exists S: \) a finite set of finite places such that

\[
K_2(O_F) \subset K_2^S(F).
\]

Thus \( K_2(O_F) \) is finitely generated. It is known that the groups is torsion. It follows from these fact that \( K_2(O_F) \) is a finite abelian group.
Computation of tame kernels

Tame kernel $K_2(O_F)$ of a number field $F$ is computable in theory:

- its order
- its structure

cf. a practical algorithm is given by Belabas-Gangl [BG04].

If $F$ is a real abelian field, the order of $K_2(O_F)$ is given by the formula (Birch-Tate conjecture, proved by Mazur-Wiles, Kolster)

$$\#K_2(O_F) = (-1)^{[F:Q]}w_2(F)\zeta_F(-1),$$

where $w_2(F) := \max\{n | \exp(\text{Gal}(F(\zeta_n)/F)) \leq 2\}$. 
Distribution of (odd parts of) tame kernels of quadratic fields

\[ F = \mathbb{Q}(\sqrt{D}) : \text{a quadratic field of the discriminant } D, \]
\[ O_D : \text{its ring of integers,} \]
\[ p : \text{an odd prime (fix),} \]

**Problem:** For a positive real number \( X \), estimate the number

\[ \# \{0 < |D| < X | p \nmid \#K_2(O_D) \} \]

in terms of \( X \).

If \( D > 0 \), one can obtain some estimate by using Birch-Tate conjecture ([Kim07]).
Distribution of (odd parts of) tame kernels of quadratic fields (cont’d)

With the same notations,

**Problem:** For a positive real number $X$, estimate the number

$$\#\{0 < |D| < X \mid p \mid \#K_2(O_D)\}$$

in terms of $X$.

(For $p = 3$, if $d > 0$ and $d \equiv 6 \pmod{9}$ then $3 \mid \#K_2(O_d)$, by J. Browkin [Bro00].

For $p = 5$, if $d > 0$, $5 \mid h(Q(\sqrt{5d}))$ then $5 \mid \#K_2(O_d)$ by [Bro92].

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Just after my talk, Prof. Y. Kishi kindly noticed me that one can deduce, from Ichimura [Ich03], there are infinitely many real quadratic fields whose class numbers and discriminants both divisible by 5. Thus we see $\exists \infty D > 0$ such that $5 \mid \#K_2(O_D)$. This has been shown already by Kimura [Kim06].
2-power ranks for finite abelian groups

Notation. $G$: a finite abelian group,

$2^i$-rank $e_i(G)$ of $G$ ($i = 1, 2, \ldots$) are defined by

$$e_i(G) = \dim_{\mathbb{Z}/2\mathbb{Z}}(G^{2^{i-1}}/G^{2^i}).$$
Distribution of (2-parts of) tame kernels of quadratic fields

Today’s theme: We want to know the following density of prime numbers $q$:

$D$: A square free integer (fix).

$e$: A natural number (fix).

$T$: A finite abelian 2-group of exponent dividing $2^e$ (fix).

$$\frac{\#\{q\mid K_2(O_{Dq})/K_2(O_{Dq})^{2^e} \cong T\}}{\#\{\text{all primes}\}} = ?,$$

where $O_{Dq}$ is the ring of integers of $\mathbb{Q}(\sqrt{Dq})$. 
Model Case: 2-part of ideal class groups

$D$: A square free integer (fix).

$e$: A natural number (fix).

$T$: A finite abelian 2-group of exponent dividing $2^e$ (fix).

$$\frac{\#\{q \mid \text{Cl}(O_{Dq})/\text{Cl}(O_{Dq})^{2^e} \cong T\}}{\#\{\text{all primes}\}} = ?,$$

where $\text{Cl}(O_{Dq})$ is the ideal class group of $\mathbb{Q}(\sqrt{Dq})$.

In some cases, the RHS is known!
Model Case: Governing field for 2-part of ideal class groups

**Fact.** (Stevenhagen [Ste89], Morton [Mor82],...) For a square free integer $D$ (with some assumptions), there is a Galois extension $\Sigma(D)/\mathbb{Q}$ such that the following equivalence holds: for a triple of integers $\rho$, $s$ and $r$ ($0 \leq \rho \leq s \leq r$),

$$
\text{Cl}(O_{Dq})/\text{Cl}(O_{Dq})^8 \cong (\mathbb{Z}/2\mathbb{Z})^{r-s} \oplus (\mathbb{Z}/4\mathbb{Z})^{\rho} \oplus (\mathbb{Z}/8\mathbb{Z})^{s-\rho}
$$

$$
\iff \left[ \Sigma(D)/\mathbb{Q}_q \right] \subset \text{Conjugacy classes depending on } \rho, s \text{ and } r.
$$

Chebotarev density theorem provides the density of such $q$. 
Governing field for ideal class group
(Cohn-Lagarias) [CL83]

This kind of phenomenon was first suggested by Cohn-Lagarias 1983.
Governing field for ideal class group

(Morton) [Mor82]

Suppose \( D = p_1 \ldots p_r, \ p_i \equiv 1 \pmod{4}, \ \left( \frac{p_i}{p_j} \right) = 1 \) for \( i \neq j, \ q \equiv 3 \pmod{4}, \)

then, \( \exists \Sigma(-D) \) such that \( \left[ \frac{\Sigma(-D)/Q}{q} \right] \) determines \( \text{Cl}(-Dq)/\text{Cl}(-Dq)^8. \)

Further, Morton shows that, in this case, \( [\Sigma(-D) : Q] = 2^{r_2+2r} \) and gave explicit density.

\( (\text{cf. Hokuriku Number Theory Workshop 2007.}) \)
Governing field for ideal class group
(Stevenhagen) [Ste89]

For any $D \in \mathbb{Z}$, $D \not\equiv 2 \pmod{4}$,

$$K_D := \mathbb{Q}(\sqrt{p^*}; p^* \mid D),$$

where $p^*$ is a prime fundamental discriminant.

$\Omega(D) :=$ the maximal abelian extension of $K_D$ unramified outside $2D\infty$ and of exponent 2 over $K_D$.

Then, $\text{Cl}(Dq)/\text{Cl}(Dq)^8$ is determined by $\left[ \frac{\Omega(D)/\mathbb{Q}}{q} \right]$.

(the most general up to now, but less explicit).
Morton’s strategy

2-rank of ideal class groups of quadratic fields $\mathbb{Q}(\sqrt{Dq})$...well known, 4-rank and 8-rank are described by certain square matrix over $\mathbb{Z}/2\mathbb{Z}$. Its entries are of the form

$\left( \frac{Np_i}{p_j}, -Dq \right)'$,

where ' means that $1' = 0 \pmod{2}$, $-1' = 1 \pmod{2}$.

Strategy: decompose the matrix into the part depends only on $D$ and depends on $q$. 

4-rank formula of $K_2$

Hurrelbrink and Kolster [HK98, lemma 5.1].

For an imaginary quadratic field $\mathbb{Q}(\sqrt{d})$, $d < 0$,

$$e_2(K_2(O_d)) = \#\{p > 2; \ p \mid d, \} - \text{rank}_{\mathbb{Z}/2\mathbb{Z}}(M(d)),$$

where $M(d)$ is the matrix of the form...
### 4-rank formula of $K_2$ (cont’d)

\[
M(d) = \begin{pmatrix}
(-d, p_1)_2 & (-d, p_1)_{p_1} & \cdots & (-d, p_1)_{p_t} \\
(-d, p_2)_2 & (-d, p_2)_{p_1} & \cdots & (-d, p_2)_{p_t} \\
\vdots & \vdots & \ddots & \vdots \\
(-d, p_{t-1})_2 & (-d, p_{t-1})_{p_1} & \cdots & (-d, p_{t-1})_{p_t} \\
(-d, v)_2 & (-d, v)_{p_1} & \cdots & (-d, v)_{p_t} \\
(-d, -1)_2 & (-d, -1)_{p_1} & \cdots & (-d, -1)_{p_t}
\end{pmatrix}^t
\]

\[
v = 2 \text{ if } 2 \not\in N(\mathbb{Q}(\sqrt{d})^\times), \quad v = u + w \text{ if } 2 \in N(\mathbb{Q}(\sqrt{d})^\times) \text{ (in this case, } d \in N(\mathbb{Q}(\sqrt{2})^\times), \text{ so } d = u^2 - 2w^2).\n\]

(Note that trailing $'$, this is a matrix over $\mathbb{Z}/2\mathbb{Z}$).
4-rank of $K_2$ for certain quadratic fields

Conner-Hurrelbrink [CH89] determined 4-ranks of $K_2$ for the following cases:

\[
\begin{align*}
  d &= pl, \quad 4 - \text{rank} = 1 \text{ or } 2, \\
  d &= 2pl, \quad 4 - \text{rank} = 1 \text{ or } 2, \\
  d &= -pl, \quad 4 - \text{rank} = 0 \text{ or } 1, \\
  d &= -2pl, \quad 4 - \text{rank} = 0 \text{ or } 1.
\end{align*}
\]

Method: Hurrelbrink-Kolster’s 4-rank formula, relation between the rank of the matrix $M(dl)$ and splitting of $l$ in certain number field, and representation of power of $l$ by positive definite binary quadratic forms.
Osburn’s computation of 4-rank densities [Osb02]

R. Osburn computed the 4-rank densities for $D = pl, 2pl, -pl, -2pl$.

$$\mathcal{L} = \left\{ l \in \mathbb{Z} \mid l \text{ is prime, } l \equiv 1 \pmod{8}, \left( \frac{l}{p} \right) = \left( \frac{p}{l} \right) = 1 \right\}$$

**Theorem** (Osburn) For the fields $\mathbb{Q}(\sqrt{pl}), \mathbb{Q}(\sqrt{2pl})$, 4-rank 1 and 2 each appear with natural density $1/2$ in $\mathcal{L}$.

For the fields $\mathbb{Q}(\sqrt{-pl}), \mathbb{Q}(\sqrt{-2pl})$, 4-rank 0 and 1 each appear with natural density $1/2$ in $\mathcal{L}$.

**Method**: a construction of a governing field (no reference to this term, though).
**4-rank formula revisited**

The formula is of the form

\[ e_2(K_2(O_{Dq})) = t - \text{rank}_{\mathbb{Z}/2\mathbb{Z}} M(Dq). \]

If one can state the condition "If \( q \) is decomposed in certain way in a certain number field, then the rank of \( M(Dq) \) is the same for those \( q \)”, then the 4-rank is the same for those \( q \).

(This gives an estimate of density of \( q \) from below.)
4-rank formula revisited (cont’d)

On the other hand, if one wants to compute the density of \( q \) which satisfies \( e_2(K_2(O_{Dq})) = e \) (\( e \) given), one must enumerate possible \( M(Dq) \).

(As Morton did in the ideal class groups case).
Conclusion

- Governing field is interesting notion (there also is a notion "Chebotarev set").
- Construction of a governing field for $K_2(O_{Dq})$ has established only for a few case (the case $D$ having a few prime factors).
- 8-rank of $K_2(O_{Dq})$?...seems difficult. cf. Vazzana [Vaz99].
References


