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Contact metric structures with the typical contact form on the 3-dimensional manifold

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Abstract. Let η be a typical contact form on the manifold $M^3 = S^3, \mathbf{R^3}$ and T^3. We determine contact metric structures (φ, ξ, η, g) on M^3. And then we consider the cases that (φ, ξ, η, g) is η-Einstein, or it is Sasakian, or it is K-contact, respectively.

1. Introduction

Given a contact form η on a C^∞ manifold M^3, it is well known that there exists a unique vector field ξ satisfying (2.1) and (2.2). Moreover, there exists a pair (g, φ) of a Riemannian metric g and a tensor field φ of type (1,1) that satisfy (2.3), (2.4) and (2.5).

(φ, ξ, η, g) is called a contact metric structure. Although ξ is unique, g and φ are not necessarily unique.

We consider a typical contact form η on $M^3 = S^3$, $\mathbf{R^3}$ and T^3 respectively. And we shall find contact metric structures with the fixed contact form η on S^3, $\mathbf{R^3}$ and T^3.

We completely determined such contact metric structure in Proposition 3.2, Proposition 4.2 and Proposition 5.3. Remark that there are a lot of contact metric structures on S^3, $\mathbf{R^3}$ and T^3 respectively.

Next, we check that such contact metric structures (φ, ξ, η, g) are η-Einstein or not, Sasakian or not, K-contact or not, respectively. In case of

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$M^3 = S^3$, (φ, ξ, η, g) is η-Einstein if and only if g is the standard metric, (φ, ξ, η, g) is Sasakian if and only if g is the standard metric and (φ, ξ, η, g) is K-contact if and only if g is the standard metric.

In case of $M^3 = \mathbf{R^3}$, all (φ, ξ, η, g) are η-Einstein, Sasakian and K-contact.

In case of $M^3 = T^3$, one parameter family of (φ, ξ, η, g) are η-Einstein, no (φ, ξ, η, g) is Sasakian and no (φ, ξ, η, g) is K-contact.

2. Preliminaries

Definition 2.1. *A $(2n+1)$-dimensional C^∞ manifold M is said to be a contact manifold if it carries a 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ holds.*

The 1-form η is called a contact form. And it is well known that there exists the unique vector field ξ satisfying

$$\eta(\xi) = 1, \tag{2.1}$$

$$d\eta(\xi, X) = 0 \quad \text{for } X, Y \in \mathfrak{X}(M). \tag{2.2}$$

The pair (M, η) is called a contact manifold and the vector field ξ is called the characteristic vector field of η.

A Riemannian metric g is said to be an associated metric if there exists a tensor field φ of type (1,1) satisfying

$$d\eta(X, Y) = g(X, \varphi Y), \tag{2.3}$$

$$\eta(X) = g(X, \xi), \tag{2.4}$$

$$\varphi^2 = -I + \eta \otimes \xi. \tag{2.5}$$

The structure (φ, ξ, η, g) is called a contact metric structure and a manifold M^{2n+1} with a contact metric structure (φ, ξ, η, g) is said to be a contact metric manifold.

Remark that, in this paper, $d\eta$ is defined by

$$d\eta(X, Y) = \frac{1}{2}(X\eta(Y) - Y\eta(X) - \eta([X, Y])) \quad \text{for } X, Y \in \mathfrak{X}(M). \tag{2.6}$$

In terms of a local coordinates $x^1, \cdots, x^{2n+1}$ of M^{2n+1}, if $\eta = \sum_{i=1}^{2n+1} \eta_i dx^i$, then $d\eta$ is expressed as

$$d\eta = \frac{1}{2} \sum_{i,j=1}^{2n+1} \frac{\partial \eta_i}{\partial x^j} dx^j \wedge dx^i. \tag{2.7}$$

We denote by ∇ the Riemannian connection of g and by R the Riemannian curvature tensor, which is defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \qquad \text{for } X, Y, Z \in \mathfrak{X}(M^{2n+1}).$$

The Ricci tensor $Ric(X,Y)$ is defined by

$$Ric(X,Y) = \sum_{i=1}^{2n+1} g(R(X_i, X)Y, X_i) \qquad \text{for } X, Y \in \mathfrak{X}(M^{2n+1}), \tag{2.8}$$

where $X_1, \cdots, X_{2n+1}$ is a local orthonormal frame field of M^{2n+1}. The Ricci operator Q is defined by

$$Ric(X,Y) = g(QX,Y) \qquad \text{for } X, Y \in \mathfrak{X}(M^{2n+1}). \tag{2.9}$$

Definition 2.2. *A contact metric structure is said to be η-Einstein if*

$$Q = pI + q\eta \otimes \xi \tag{2.10}$$

holds, where p, q are some smooth functions on M^{2n+1}.

Remark that (2.10) is equivalent to

$$Ric(X,Y) = pg(X,Y) + qg(\xi, X)g(\xi, Y) \qquad \text{for } X, Y \in \mathfrak{X}(M^{2n+1}). \tag{2.11}$$

Definition 2.3. *A contact metric manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$ is said to be Sasakian if M^{2n+1} satisfies*

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y \qquad \textit{for } X, Y \in \mathfrak{X}(M^{2n+1}). \tag{2.12}$$

Definition 2.4. *A contact metric structure (φ, ξ, η, g) is called a K-contact if ξ is a Killing vector field, i.e., $\mathcal{L}_\xi g = 0$.*

Remark that $\mathcal{L}_\xi g = 0$ is equivalent to

$$g(X, \nabla_Y \xi) + g(\nabla_X \xi, Y) = 0 \qquad \text{for } X, Y \in \mathfrak{X}(M^{2n+1}). \tag{2.13}$$

3. S^3 with the contact form η

Let $(x^1, \cdots, x^{2n+2})$ be Cartesian coordinates on the (2n+2)-dimensional Euclidean space $\mathbf{R}^{2n+2}$. We consider the 1-form α on $\mathbf{R}^{2n+2}$ defined by

$$\alpha = x^1 dx^2 - x^2 dx^1 + \cdots + x^{2n+1} dx^{2n+2} - x^{2n+2} dx^{2n+1} \tag{3.1}$$

and the inclusion mapping

$$\iota : S^{2n+1} \to \mathbf{R}^{2n+2}. \tag{3.2}$$

It is well known that $\eta = \iota^* \alpha$ is a contact form on S^{2n+1}, i.e., $\eta \wedge (d\eta)^n \neq 0$ holds on S^{2n+1}. By using (3.1), from (2.7) we get

$$d\alpha = dx^1 \wedge dx^2 + dx^3 \wedge dx^4 + \cdots + dx^{2n+1} \wedge dx^{2n+2}. \tag{3.3}$$

Throughout this section, we consider this contact form η on S^3. Then from (2.1) and (2.2), the characteristic vector field ξ is determined by

$$d\iota(\xi) = -x^2 \frac{\partial}{\partial x^1} + x^1 \frac{\partial}{\partial x^2} - x^4 \frac{\partial}{\partial x^3} + x^3 \frac{\partial}{\partial x^4}.$$

We take the independent vector fields $X_1, X_2, X_3 = \xi$ on S^3 such that

$$d\iota(X_1) = -x^3 \frac{\partial}{\partial x^1} + x^4 \frac{\partial}{\partial x^2} + x^1 \frac{\partial}{\partial x^3} - x^2 \frac{\partial}{\partial x^4}, \tag{3.4}$$

$$d\iota(X_2) = -x^4 \frac{\partial}{\partial x^1} - x^3 \frac{\partial}{\partial x^2} + x^2 \frac{\partial}{\partial x^3} + x^1 \frac{\partial}{\partial x^4}, \tag{3.5}$$

$$d\iota(X_3) = -x^2 \frac{\partial}{\partial x^1} + x^1 \frac{\partial}{\partial x^2} - x^4 \frac{\partial}{\partial x^3} + x^3 \frac{\partial}{\partial x^4}. \tag{3.6}$$

3.1. g and φ of S^3

Let g be a Riemannian metric on (S^3, η) which satisfies (2.4). We put $g_{ij} = g(X_i, X_j)$ and $a = g_{11},\ b = g_{12} = g_{21},\ c = g_{22}$.

By using $\eta = \iota^* \alpha$, from (2.4) we get

$$g_{13} = g(X_1, X_3) = \eta(X_1) = 0,$$

$$g_{23} = g(X_2, X_3) = \eta(X_2) = 0$$

and from (2.1) get

$$g_{33} = g(X_3, X_3) = \eta(X_3) = 1.$$

Then, the 3×3 matrix (g_{ij}) is of the form

$$(g_{ij}) = \begin{pmatrix} a & b & 0 \\ b & c & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.7}$$

where $a, b, c \in C^\infty(S^3)$.

Since $\det(g_{ij}) > 0$, we get $ac - b^2 > 0$. Moreover, since $X_1 \neq 0, X_2 \neq 0$, we get $a = g(X_1, X_1) > 0, c = g(X_2, X_2) > 0$.

Conversely, let g be a tensor field of type (0,2) defined by (3.7). If $a > 0, c > 0$ and $ac - b^2 > 0$ holds, then g is a Riemannian metric satisfying (2.4).

Thus we have the following.

Proposition 3.1. *If a Riemannian metric g on (S^3, η) satisfies (2.4), then (3.7) and the following hold*

$$a > 0, c > 0 \quad \textit{and} \quad ac - b^2 > 0. \tag{3.8}$$

Conversely, let g be a tensor field of type (0,2) on (S^3, η) defined by (3.7). If g satisfies (3.8), then g is a Riemannian metric on (S^3, η) and satisfies (2.4).

Next, let φ be a tensor field of type (1,1) satisfying (2.3). Then, we have

$$\begin{aligned} \varphi(X_1) &= \frac{b}{ac - b^2} X_1 + \frac{-a}{ac - b^2} X_2, \\ \varphi(X_2) &= \frac{c}{ac - b^2} X_1 + \frac{-b}{ac - b^2} X_2, \\ \varphi(X_3) &= 0, \end{aligned}$$

where $a > 0, c > 0, ac - b^2 > 0$.
Because, by using $\eta = \iota^* \alpha$ from (3.3) we get

$$d\eta(X_i, X_j) = (dx^1 \wedge dx^2 + dx^3 \wedge dx^4)(d\iota(X_i), d\iota(X_j)).$$

And then from (3.4), (3.5), (3.6) we have

$$d\eta(X_1, X_2) = 1, \ d\eta(X_2, X_1) = -1, \ \text{others are equal to } 0.$$

Now, we put $\varphi(X_j) = \sum_{k=1}^{3} \varphi_{kj} X_k \quad (j = 1, 2, 3)$. Since $g(X_i, \varphi X_j) = \sum_{k=1}^{3} g_{ik}\varphi_{kj}$, from (2.3) we get

$$(g_{ij})(\varphi_{ij}) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus

$$\begin{pmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ \varphi_{21} & \varphi_{22} & \varphi_{23} \\ \varphi_{31} & \varphi_{32} & \varphi_{33} \end{pmatrix} = \frac{1}{ac - b^2} \begin{pmatrix} b & c & 0 \\ -a & -b & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.9}$$

where $a > 0, c > 0, ac - b^2 > 0$.

Proposition 3.2. *Let (φ, ξ, η, g) be given by (3.7), (3.8) and (3.9) on S^3. If (φ, ξ, η, g) is a contact metric structure, then the following equation holds*

$$ac - b^2 = 1. \tag{3.10}$$

Conversely, if (3.10) holds, then (φ, ξ, η, g) is a contact metric structure on S^3.

Proof. From (3.9) we get

$$(\varphi_{ij})^2 = \frac{1}{b^2 - ac} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.11}$$

By putting $\psi = -I + \eta \otimes \xi$, we get the following equation

$$\psi(X_j) = -X_j + \eta(X_j)X_3 \qquad (j = 1, 2, 3).$$

By substituting $j = 1, 2, 3$ into the above equation, we get

$$\begin{aligned} \psi(X_1) &= -X_1 + \eta(X_1)X_3 = -X_1, \\ \psi(X_2) &= -X_2 + \eta(X_2)X_3 = -X_2, \\ \psi(X_3) &= -X_3 + \eta(X_3)X_3 = 0. \end{aligned}$$

Now, we put

$$\psi(X_j) = \psi_{1j}X_1 + \psi_{2j}X_2 + \psi_{3j}X_3 = \sum_{i=1}^{3} \psi_{ij}X_i.$$

By substituting $j = 1, 2, 3$ into the above equation, from the above result we get

$$\begin{pmatrix} \psi_{11} & \psi_{12} & \psi_{13} \\ \psi_{21} & \psi_{22} & \psi_{23} \\ \psi_{31} & \psi_{32} & \psi_{33} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.12}$$

If (φ, ξ, η, g) is a contact metric structure, by using (3.11) and (3.12) from (2.5) we get (3.10).

Conversely, if (3.10) holds, we can get (2.5). This completes the proof. □

Corollary 3.1. *φ is denoted by the following matrix*

$$\begin{pmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ \varphi_{21} & \varphi_{22} & \varphi_{23} \\ \varphi_{31} & \varphi_{32} & \varphi_{33} \end{pmatrix} = \begin{pmatrix} b & c & 0 \\ -a & -b & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ a > 0, c > 0, \ ac - b^2 = 1.$$

3.2. Curvature tensors

In this section, we assume a, b, c are constant. By using (g_{ij}) which satisfies (3.7),(3.8) and (3.10), from the basis $X_1, X_2, X_3 = \xi$, we can generate the orthonormal basis Y_1, Y_2, Y_3 on (S^3, g), that is

$$Y_1 = X_3, \ Y_2 = \frac{1}{\sqrt{a}}X_1, \ Y_3 = -\frac{\sqrt{a}b}{a}X_1 + \sqrt{a}X_2.$$

And then we get

$$X_3 = Y_1, \ X_1 = \sqrt{a}Y_2, \ X_2 = \frac{b}{\sqrt{a}}Y_2 + \frac{1}{\sqrt{a}}Y_3.$$

By computing $[X_i, X_j]$ from (3.4), (3.5), (3.6) and the above equations, we have

$$[X_1, X_2] = -2X_3, \ [X_2, X_3] = -2X_1, \ [X_3, X_1] = -2X_2$$

and

$$[Y_1,Y_2]=-\frac{2b}{a}Y_2-\frac{2}{a}Y_3,\ [Y_1,Y_3]=\frac{2(a^2+b^2)}{a}Y_2+\frac{2b}{a}Y_3,\ [Y_2,Y_3]=-2Y_1.$$

From the above results, we get

$$\begin{array}{ll}
2g(\nabla_{Y_2}Y_2,Y_1)=-\frac{4b}{a}, & 2g(\nabla_{Y_3}Y_3,Y_1)=\frac{4b}{a},\\
2g(\nabla_{Y_1}Y_2,Y_3)=\frac{2(a-a^2-b^2-1)}{a}, & 2g(\nabla_{Y_1}Y_3,Y_2)=\frac{2(a^2+b^2-a+1)}{a},\\
2g(\nabla_{Y_2}Y_3,Y_1)=\frac{2(a^2+b^2-a-1)}{a}, & 2g(\nabla_{Y_2}Y_1,Y_2)=\frac{4b}{a},\\
2g(\nabla_{Y_2}Y_1,Y_3)=\frac{2(a-a^2-b^2+1)}{a}, & 2g(\nabla_{Y_3}Y_1,Y_2)=\frac{2(1-a-a^2-b^2)}{a},\\
2g(\nabla_{Y_3}Y_1,Y_3)=-\frac{4b}{a}, & 2g(\nabla_{Y_3}Y_2,Y_1)=\frac{2(a^2+b^2+a-1)}{a}.
\end{array}$$

others are equal to 0.

And then we have

$$\begin{array}{ll}
\nabla_{Y_1}Y_1=0, & \nabla_{Y_1}Y_2=\frac{a-a^2-b^2-1}{a}Y_3,\\
\nabla_{Y_1}Y_3=\frac{-a+a^2+b^2+1}{a}Y_2, & \nabla_{Y_2}Y_1=\frac{2b}{a}Y_2+\frac{a-a^2-b^2+1}{a}Y_3,\\
\nabla_{Y_2}Y_2=-\frac{2b}{a}Y_1, & \nabla_{Y_2}Y_3=\frac{-a+a^2+b^2-1}{a}Y_1,\\
\nabla_{Y_3}Y_1=\frac{-a-a^2-b^2+1}{a}Y_2-\frac{2b}{a}Y_3, & \nabla_{Y_3}Y_2=\frac{a+a^2+b^2-1}{a}Y_1,\\
\nabla_{Y_3}Y_3=\frac{2b}{a}Y_1. &
\end{array}$$

Next, we put

$$\alpha = 1-a-c, \tag{3.13}$$

$$\beta = \frac{2b}{a}, \tag{3.14}$$

$$\gamma = \alpha+\frac{2}{a}. \tag{3.15}$$

By using the above equations and (3.10), we have

$$[Y_1,Y_2]=-\beta Y_2-(\gamma-\alpha)Y_3,\ [Y_1,Y_3]=(2-\alpha-\gamma)Y_2+\beta Y_3,\ [Y_2,Y_3]=-2Y_1,$$

and

$$\begin{array}{lll}
\nabla_{Y_1}Y_1=0, & \nabla_{Y_1}Y_2=\alpha Y_3, & \nabla_{Y_1}Y_3=-\alpha Y_2,\\
\nabla_{Y_2}Y_1=\beta Y_2+\gamma Y_3, & \nabla_{Y_2}Y_2=-\beta Y_1, & \nabla_{Y_2}Y_3=-\gamma Y_1,\\
\nabla_{Y_3}Y_1=(\gamma-2)Y_2-\beta Y_3, & \nabla_{Y_3}Y_2=-(\gamma-2)Y_1, & \nabla_{Y_3}Y_3=\beta Y_1.
\end{array}$$

Hence we have

$$\begin{aligned}
R(Y_1,Y_2)Y_1 &= (\alpha^2-2\gamma\alpha-4)Y_2+2\alpha\beta Y_3,\\
R(Y_1,Y_2)Y_2 &= (-\alpha^2+2\gamma\alpha+4)Y_1,\\
R(Y_1,Y_2)Y_3 &= -2\alpha\beta Y_1,\\
R(Y_1,Y_3)Y_1 &= 2\alpha\beta Y_2+(\alpha^2+2\gamma\alpha-4\alpha-4)Y_3,\\
R(Y_1,Y_3)Y_2 &= -2\alpha\beta Y_1,\\
R(Y_1,Y_3)Y_3 &= (-\alpha^2-2\gamma\alpha+4\alpha+4)Y_1,\\
R(Y_2,Y_3)Y_1 &= 0,\\
R(Y_2,Y_3)Y_2 &= (-\alpha^2+4\alpha+4)Y_3,\\
R(Y_2,Y_3)Y_3 &= (\alpha^2-4\alpha-4)Y_2.
\end{aligned}$$

From the above result, by using (2.8) we get

$$(3.16)\quad \big(Ric(Y_i,Y_j)\big)=\begin{pmatrix}-2\alpha^2+4\alpha+8 & 0 & 0\\ 0 & 2\gamma\alpha-4\alpha & -2\alpha\beta\\ 0 & -2\alpha\beta & -2\gamma\alpha\end{pmatrix}$$

$$=\begin{pmatrix}-2(a+c)^2+10 & 0 & 0\\ 0 & 2(a+c-1)(a+c+1-\frac{2}{a}) & \frac{4b}{a}(a+c-1)\\ 0 & \frac{4b}{a}(a+c-1) & -2(a+c-1)(a+c-1-\frac{2}{a})\end{pmatrix}.$$

Proposition 3.3. *Let (S^3,φ,ξ,η,g) be the contact metric manifold determined by Proposition 3.2 and we assume that a,b,c are constant.*

(1) (S^3,φ,ξ,η,g) is η-Einstein if and only if $b=0, a=c=1$, that is, (S^3,η,ξ,g,φ) is the standard 3-dimensional sphere.

(2) (S^3,φ,ξ,η,g) is Sasakian if and only if $b=0, a=c=1$, that is, (S^3,φ,ξ,η,g) is the standard 3-dimensional sphere.

(3) (S^3,φ,ξ,η,g) is K-contact if and only if $b=0, a=c=1$, that is, (S^3,φ,ξ,η,g) is the standard 3-dimensional sphere.

Proof. (1) If S^3 is η-Einstein, by substituting $(Y_i,Y_j)=(Y_1,Y_1),(Y_2,Y_2)$ into (2.11), from (3.16) we get

$$\begin{aligned}(3.17)\quad Ric(Y_i,Y_j) &= 2\alpha(\gamma-2)g(Y_i,Y_j)\\ &\quad +2(-\alpha^2-\gamma\alpha+4\alpha+4)g(Y_1,Y_i)g(Y_1,Y_j).\end{aligned}$$

Moreover, we substitute $(Y_i, Y_j) = (Y_1, Y_2), (Y_1, Y_3), (Y_2, Y_3), (Y_3, Y_3)$ into (3.17) and hence get

$$\alpha \neq 0, \quad \beta = 0, \text{ i.e., } b = 0, \quad a = c = 1.$$

Conversely, if $b = 0, \quad a = c = 1$, (3.17) holds.
(2) If S^3 is Sasakian, by substituting $(X, Y) = (Y_1, Y_2), (Y_1, Y_3)$ into (2.12) we get

$$\begin{aligned} \alpha\beta &= 0, \\ \alpha^2 - 2\gamma\alpha - 4 &= -1, \\ \alpha^2 + 2\gamma\alpha - 4\alpha - 4 &= -1. \end{aligned}$$

Therefore, we get

$$\alpha \neq 0, \quad \beta = 0, \text{ i.e., } b = 0, \quad a = c = 1.$$

Conversely, if $b = 0, \quad a = c = 1$, (2.12) holds.
(3) If S^3 is K-contact, by substituting $(X, Y) = (Y_i, Y_j)$ into (2.13), we get

$$\begin{aligned} \frac{4b}{a} &= 0, \\ \frac{2(-a^2 - b^2 + 1)}{a} &= 0. \end{aligned}$$

And then we have

$$b = 0, \quad a = c = 1.$$

Conversely, if $b = 0$, $a = c = 1$, (2.13) holds. This completes the proof. □

Remark. If (S^3, g) is a contact metric manifold which does not satisfy $b = 0,\ a = c = 1$, then (S^3, g) is neither η-Einstein nor Sasakian, K-contact.

4. $\mathbf{R}^3$ with the contact form η

Let η be the 1-form on $\mathbf{R}^3$ defined by

$$\eta = \frac{1}{2}(dx^3 - x^2 dx^1). \tag{4.1}$$

From (2.7) we get

$$\eta \wedge d\eta = \frac{1}{8}(dx^1 \wedge dx^2 \wedge dx^3) \neq 0, \tag{4.2}$$

i.e., η is a contact form on $\mathbf{R}^3$.

From (2.1), (2.2) we get

$$\xi = 2\frac{\partial}{\partial x^3}. \tag{4.3}$$

4.1. g and φ of $\mathbf{R}^3$

Let g be a Riemannian metric on $(\mathbf{R}^3, \eta)$ which satisfies (2.4). We put $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$ and $a = g_{11},\ b = g_{12} = g_{21},\ c = g_{22}$. By using (4.1) and (4.3), from (2.4) we have the following matrix

$$(g_{ij}) = \begin{pmatrix} a & b & -\frac{1}{4}x^2 \\ b & c & 0 \\ -\frac{1}{4}x^2 & 0 & \frac{1}{4} \end{pmatrix}, \tag{4.4}$$

where $a, b, c \in C^\infty(\mathbf{R}^3)$.

Since $\det(g_{ij}) > 0$, we get

$$(a - \frac{1}{4}(x^2)^2)c - b^2 > 0. \tag{4.5}$$

Moreover, since $\frac{\partial}{\partial x^1} \neq 0$ and $\frac{\partial}{\partial x^2} \neq 0$, we get $a = g(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^1}) > 0, c = g(\frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^2}) > 0$.

Conversely, let g be a tensor field of type (0,2) defined by (4.4). If $a > 0, c > 0$ and (4.5) hold, then we get $g_{11} > 0, g_{22} > 0, \det(g_{ij}) > 0$ and hence

$$ac - b^2 > 0. \tag{4.6}$$

And then g is a Riemannian metric satisfying (2.4).

Because, let λ be an eigenvalue of (g_{ij}), λ satisfies the following equation

$$\begin{aligned} &16\lambda^3 - 4(4a + 4c + 1)\lambda^2 + (4a + 4c + 16ac - 16b^2 - (x^2)^2)\lambda \\ &\quad + c(x^2)^2 - 4(ac - b^2) = 0. \end{aligned} \tag{4.7}$$

We put the left side of (4.7) by $f(\lambda)$. Then from (4.5) we have

$$f(0) = c(x^2)^2 - 4(ac - b^2) < 0. \tag{4.8}$$

The differential of $f(\lambda)$ is

$$f'(\lambda) = 48\lambda^2 - 8(4a + 4c + 1)\lambda + (4a + 4c + 16ac - 16b^2 - (x^2)^2).$$

On the other hand, by using (4.5) and (4.6), we get

$$4a + 4c + 16ac - 16b^2 > \frac{4(ac - b^2)}{c} > (x^2)^2. \tag{4.9}$$

Therefore, if a discriminant of the quadratic equation $f'(\lambda) = 0$ of λ is non-negative, from (4.9) $f'(\lambda) = 0$ has a positive solution. And hence from (4.8), λ is a positive number. Also, if a discriminant of $f'(\lambda) = 0$ is negative, from (4.8) λ is a positive number. Moreover, we can see that g satisfies (2.4).

Thus we have the following.

Proposition 4.1. *If a Riemannian metric g on $(\mathbf{R}^3, \eta)$ satisfies (2.4), then (4.4) and the following holds*

$$a > 0, \ c > 0, \quad (a - \frac{1}{4}(x^2)^2)c - b^2 > 0. \tag{4.10}$$

Conversely, let g be a tensor field of type (0,2) on $(\mathbf{R}^3, \eta)$ defined by (4.4). If g satisfies (4.10), then g is a Riemannian metric on $(\mathbf{R}^3, \eta)$ and satisfies (2.4).

Next, we denote the left side of (4.5) by G, i.e.,

$$(a - \frac{1}{4}(x^2)^2)c - b^2 = G. \tag{4.11}$$

Let φ be a tensor field of type (1,1) satisfying (2.3) and put

$$\varphi(\frac{\partial}{\partial x^j}) = \sum_{k=1}^{3} \varphi_{kj} \frac{\partial}{\partial x^k} \ (j = 1, 2, 3).$$

Corollary 4.1. *If (g_{ij}) defined by (4.4) satisfies (4.10), then*

$$\begin{pmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ \varphi_{21} & \varphi_{22} & \varphi_{23} \\ \varphi_{31} & \varphi_{32} & \varphi_{33} \end{pmatrix} = \frac{1}{4G} \begin{pmatrix} b & c & 0 \\ -a + \frac{1}{4}(x^2)^2 & -b & 0 \\ x^2 b & x^2 c & 0 \end{pmatrix} \tag{4.12}$$

holds, where $a > 0, \ c > 0, \ (a - \frac{1}{4}(x^2)^2)c - b^2 > 0$.

Proof. By substituting $X = \dfrac{\partial}{\partial x^i}$, $Y = \dfrac{\partial}{\partial x^j}$ into (2.3), we get

$$\begin{pmatrix} a & b & -\frac{1}{4}x^2 \\ b & c & 0 \\ -\frac{1}{4}x^2 & 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ \varphi_{21} & \varphi_{22} & \varphi_{23} \\ \varphi_{31} & \varphi_{32} & \varphi_{33} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{4} & 0 \\ -\frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $\det(g_{ij}) > 0$, we get (4.12). This completes the proof. □

Proposition 4.2. *Let (φ, ξ, η, g) be given by (4.4), (4.10) and (4.12) on $\mathbf{R}^3$. If (φ, ξ, η, g) is a contact metric structure, then*

$$G = \frac{1}{16} \tag{4.13}$$

holds. Conversely, if (4.13) holds, then (φ, ξ, η, g) is a contact metric structure on $\mathbf{R}^3$.

Proof. If (φ, ξ, η, g) is a contact metric structure, then (2.5) holds. By substituting (4.12), (4.1) and (4.3) into (2.5), we get

$$\frac{1}{16G^2} \begin{pmatrix} b & c & 0 \\ -a + \frac{1}{4}(x^2)^2 & -b & 0 \\ x^2 b & x^2 c & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ -x^2 & 0 & 0 \end{pmatrix}.$$

Then we have (4.13).

Conversely, if (4.13) holds, we can get (2.5). This completes the proof. □

Corollary 4.2. *φ is denoted by the following matrix*

$$\begin{pmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ \varphi_{21} & \varphi_{22} & \varphi_{23} \\ \varphi_{31} & \varphi_{32} & \varphi_{33} \end{pmatrix} = 4 \begin{pmatrix} b & c & 0 \\ -a + \frac{1}{4}(x^2)^2 & -b & 0 \\ x^2 b & x^2 c & 0 \end{pmatrix},$$

where $a > 0,\ c > 0,\ (a - \frac{1}{4}(x^2)^2)c - b^2 > 0$.

4.2. Curvature tensors

In this section, we assume that a, b, c are constant. We put $X_1 = \xi,\ X_2 = \dfrac{\partial}{\partial x^1},\ X_3 = \dfrac{\partial}{\partial x^2}$ on $(\mathbf{R}^3, g)$. By using g that satisfies (4.4), (4.10) and (4.13), from the basis X_1, X_2, X_3 we can generate the orthonormal basis Y_1, Y_2, Y_3 on $(\mathbf{R}^3, g)$, that is

$$Y_1 = 2\frac{\partial}{\partial x^3},\ Y_2 = \alpha(\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^3}),\ Y_3 = 4(-\alpha b\frac{\partial}{\partial x^1} + \frac{1}{\alpha}\frac{\partial}{\partial x^2} - \alpha b x^2\frac{\partial}{\partial x^3}),$$

where $\alpha = \dfrac{4\sqrt{c}}{\sqrt{16b^2+1}}$. Then we have

$$[Y_1, Y_2] = 0,\ [Y_2, Y_3] = -2Y_1,\ [Y_1, Y_3] = 0.$$

Then we may see that

$$\begin{array}{lll} 2g(\nabla_{Y_1}Y_2, Y_3) = 2, & 2g(\nabla_{Y_1}Y_3, Y_2) = -2, & 2g(\nabla_{Y_2}Y_1, Y_3) = 2, \\ 2g(\nabla_{Y_2}Y_3, Y_1) = -2, & 2g(\nabla_{Y_3}Y_1, Y_2) = -2, & 2g(\nabla_{Y_3}Y_2, Y_1) = 2, \end{array}$$

and the others are equal to 0. Therefore, we have

$$\begin{array}{lllll} \nabla_{Y_1}Y_1 = 0, & \nabla_{Y_1}Y_2 = Y_3, & \nabla_{Y_1}Y_3 = -Y_2, & \nabla_{Y_2}Y_1 = Y_3, & \nabla_{Y_2}Y_2 = 0, \\ \nabla_{Y_2}Y_3 = -Y_1, & \nabla_{Y_3}Y_1 = -Y_2, & \nabla_{Y_3}Y_2 = Y_1, & \nabla_{Y_3}Y_3 = 0. & \end{array}$$

Hence we get

$$\begin{array}{lll} R(Y_1, Y_2)Y_1 = -Y_2, & R(Y_1, Y_2)Y_2 = Y_1, & R(Y_1, Y_2)Y_3 = 0, \\ R(Y_1, Y_3)Y_1 = -Y_3, & R(Y_1, Y_3)Y_2 = 0, & R(Y_1, Y_3)Y_3 = Y_1, \\ R(Y_2, Y_3)Y_1 = 0, & R(Y_2, Y_3)Y_2 = 3Y_3, & R(Y_2, Y_3)Y_3 = -3Y_2. \end{array}$$

Using (2.8), we have

$$(4.14) \qquad \big(Ric(Y_i, Y_j)\big) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Proposition 4.3. $(\mathbf{R}^3, g)$ *is η-Einstein, Sasakian and K-contact.*

Proof. Substituting $(Y_i, Y_j) = (Y_1, Y_1), (Y_2, Y_2)$ into (2.11), from (4.14), we get

$$Ric(Y_i, Y_j) = -2g(Y_i, Y_j) + 4g(Y_1, Y_i)g(Y_1, Y_j).$$

Moreover, we can see that if $(Y_i, Y_j) = (Y_1, Y_2), (Y_1, Y_3), (Y_2, Y_3)$ and (Y_3, Y_3), then the above equation holds. Therefore, $(\mathbf{R}^3, g)$ is η-Einstein.

Next, we shall check whether $(\mathbf{R}^3, g)$ satisfies (2.12), i.e.,

$$R(Y_i, Y_j)Y_1 = g(Y_1, Y_j)Y_i - g(Y_1, Y_i)Y_j,$$

for $i, j = 1, 2, 3$. From values $R(Y_i, Y_j)Y_k$ of the curvature tensor, we may see that the above equation holds. Therefore, $(\mathbf{R}^3, g)$ is Sasakian.

Finally, we shall check whether $(\mathbf{R}^3, g)$ satisfies (2.13), i.e.,

$$2g(Y_i, \nabla_{Y_j} Y_1) + 2g(\nabla_{Y_i} Y_1, Y_j) = 0 \quad (i, j = 1, 2, 3).$$

From the calculation of $2g(\nabla_{Y_i} Y_j, Y_k)$, we may see that the above equation holds. Therefore, $(\mathbf{R}^3, g)$ is K-contact. This completes the proof. □

5. T^3 with the contact form η

Let η be the 1-form on T^3 defined by

$$\eta = \cos nx^3 dx^1 + \sin nx^3 dx^2, \qquad n \in \mathbf{N}. \tag{5.1}$$

From (2.7) we get

$$\eta \wedge d\eta = -\frac{1}{2} n dx^1 \wedge dx^2 \wedge dx^3 \neq 0, \tag{5.2}$$

i.e., η is a contact form on T^3.

From (2.1), (2.2), we get

$$\xi = \cos nx^3 \frac{\partial}{\partial x^1} + \sin nx^3 \frac{\partial}{\partial x^2}. \tag{5.3}$$

Let g be a Riemannian metric on (T^3, η) which satisfies (2.4). We put $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$ and $a = g_{11}, b = g_{12} = g_{21}, c = g_{22}$.

By using (5.1) and (5.3), from (2.4) we get

$$a \cos nx^3 + b \sin nx^3 = \cos nx^3 \tag{5.4}$$

$$b \cos nx^3 + c \sin nx^3 = \sin nx^3 \tag{5.5}$$

$$g_{31} \cos nx^3 + g_{32} \sin nx^3 = 0. \tag{5.6}$$

Proposition 5.1. *(5.4), (5.5) and (5.6) hold if and only if there exist $\beta, \alpha, g_{33} \in C^\infty(T^3)$ which satisfy the following matrix (g_{ij})*

$$(g_{ij}) = \begin{pmatrix} \beta \sin^2 nx^3 + 1 & -\beta \sin nx^3 \cos nx^3 & -\alpha \sin nx^3 \\ -\beta \sin nx^3 \cos nx^3 & \beta \cos^2 nx^3 + 1 & \alpha \cos nx^3 \\ -\alpha \sin nx^3 & \alpha \cos nx^3 & g_{33} \end{pmatrix}. \tag{5.7}$$

Proof. If (5.4) and (5.5) hold, there exist $l, k \in \mathbf{R}$ which satisfy the following equations

$$a - 1 = k(-\sin nx^3), \tag{5.8}$$

$$b = k\cos nx^3, \tag{5.9}$$

$$b = l(-\sin nx^3), \tag{5.10}$$

$$c - 1 = l\cos nx^3. \tag{5.11}$$

From (5.9) and (5.10) we get

$$k\cos nx^3 = l(-\sin nx^3).$$

When $\cos nx^3 \neq 0$ and $\sin nx^3 \neq 0$ hold, we get

$$\frac{k}{-\sin nx^3} = \frac{l}{\cos nx^3}.$$

By putting $\beta = \dfrac{k}{-\sin nx^3} = \dfrac{l}{\cos nx^3}$, from (5.6) we get (5.7). Moreover, (5.7) includes the case that either $\cos nx^3 = 0$ or $\sin nx^3 = 0$ holds.

Conversely, we can see that (g_{ij}) satisfies (5.4), (5.5) and (5.6).
This completes the proof. □

5.1. g and φ of T^3

For $\beta, \alpha, g_{33} \in \mathbf{R}$ and $n \in \mathbf{N}$, we define the matrix B by

$$B = \begin{pmatrix} \beta\sin^2 nx^3 + 1 & -\beta\sin nx^3\cos nx^3 & -\alpha\sin nx^3 \\ -\beta\sin nx^3\cos nx^3 & \beta\cos^2 nx^3 + 1 & \alpha\cos nx^3 \\ -\alpha\sin nx^3 & \alpha\cos nx^3 & g_{33} \end{pmatrix}. \tag{5.12}$$

Proposition 5.2. *Let g be the tensor field of type (0,2) on (T^3, η) defined by the matrix B, where $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$ and $B = (g_{ij})$.*

g is a Riemannian metric satisfying (2.4) if and only if the following conditions hold

$$(1+\beta)g_{33} - \alpha^2 > 0\ ,\ g_{33} > 0. \tag{5.13}$$

Proof. From Proposition 5.1 g satisfies (2.4). If g is a Riemannian metric, since $\det(g_{ij}) > 0$, we get

$$\det(B) = (1+\beta)g_{33} - \alpha^2 > 0.$$

Next, we put an eigenvalue of $B = \lambda$ and $g(\lambda) = \det(B - \lambda I)$. Then, we get

$$g(\lambda) = (1-\lambda)\{\lambda^2 - (1+\beta+g_{33})\lambda + (1+\beta)g_{33} - \alpha^2\}.$$

One of solution in $g(\lambda) = 0$ is equal to 1. The other solutions are in the following equation

$$\lambda^2 - (1+\beta+g_{33})\lambda + (1+\beta)g_{33} - \alpha^2 = 0. \quad (5.14)$$

By putting a discriminant of the above equation = D, we get

$$D = \{g_{33} - (1+\beta)\}^2 + 4\alpha^2 \geq 0.$$

Since λ are positive definite, from (5.14) we get $g_{33} > 0$.

Conversely if (5.13) holds, we can see that $g_{11} > 0$, $g_{22} > 0$, $\det(B) > 0$ and an eigenvalue of B are positive definite. This completes the proof. □

Next, let φ a tensor field of type (1,1) satisfying (2.3). We put

$$\varphi(\frac{\partial}{\partial x^j}) = \sum_{k=1}^{3} \varphi_{kj} \frac{\partial}{\partial x^k} \ (j = 1,2,3).$$

Corollary 5.1. *If (g_{ij}) defined by the matrix B satisfies (5.13), then the following equation holds.*

$$\begin{pmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ \varphi_{21} & \varphi_{22} & \varphi_{23} \\ \varphi_{31} & \varphi_{32} & \varphi_{33} \end{pmatrix} = \frac{n}{2|B|} \begin{pmatrix} -\alpha \sin^2 nx^3 & \alpha \sin nx^3 \cos nx^3 & g_{33} \sin nx^3 \\ \alpha \sin nx^3 \cos nx^3 & -\alpha \cos^2 nx^3 & -g_{33} \cos nx^3 \\ -(1+\beta) \sin nx^3 & (1+\beta) \cos nx^3 & \alpha \end{pmatrix}, \quad (5.15)$$

where $(1+\beta)g_{33} - \alpha^2 > 0$, $g_{33} > 0$.

Proof. Since $\det(g_{ij}) > 0$, from (2.3) we get (5.15). □

We put

$$\rho = \det(B) = (1+\beta)g_{33} - \alpha^2. \quad (5.16)$$

Proposition 5.3. *Let* (φ, ξ, η, g) *be given by (5.12), (5.13) and (5.15) on* T^3*. If* (φ, ξ, η, g) *is a contact metric structure, then*

$$n^2 = 4\rho \tag{5.17}$$

holds. Conversely, if (5.17) holds, then (φ, ξ, η, g) *is a contact metric structure on* T^3*.*

Proof. If (φ, ξ, η, g) is a contact metric structure, by substituting (5.15), (5.1) and (5.3) into (2.5) we get

$$\frac{n^2}{4\rho}\begin{pmatrix} -\sin^2 nx^3 & \sin nx^3 \cos nx^3 & 0 \\ \sin nx^3 \cos nx^3 & -\cos^2 nx^3 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} -\sin^2 nx^3 & \sin nx^3 \cos nx^3 & 0 \\ \sin nx^3 \cos nx^3 & -\cos^2 nx^3 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Hence we get (5.17).

Conversely, if (5.17) holds, we can get (2.5). This completes the proof. □

Corollary 5.2. φ *is denoted by the following matrix*

$$\begin{pmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ \varphi_{21} & \varphi_{22} & \varphi_{23} \\ \varphi_{31} & \varphi_{32} & \varphi_{33} \end{pmatrix} = \frac{2}{n}\begin{pmatrix} -\alpha \sin^2 nx^3 & \alpha \sin nx^3 \cos nx^3 & g_{33} \sin nx^3 \\ \alpha \sin nx^3 \cos nx^3 & -\alpha \cos^2 nx^3 & -g_{33} \cos nx^3 \\ -(1+\beta) \sin nx^3 & (1+\beta)\cos nx^3 & \alpha \end{pmatrix},$$

where $(1+\beta)g_{33} - \alpha^2 > 0$, $g_{33} > 0$.

5.2. Curvature tensors

We take the following basis on (T^3, g),

$$X_1 = \xi = \cos nx^3 \frac{\partial}{\partial x^1} + \sin nx^3 \frac{\partial}{\partial x^2},\ X_2 = -\sin nx^3 \frac{\partial}{\partial x^1} + \cos nx^3 \frac{\partial}{\partial x^2},\ X_3 = \frac{\partial}{\partial x^3}.$$

By using g that satisfies (5.12),(5.13) and (5.17), from the above basis we get the following orthonormal basis Y_1, Y_2, Y_3 on (T^3, g),

$$Y_1 = \xi = \cos nx^3 \tfrac{\partial}{\partial x^1} + \sin nx^3 \tfrac{\partial}{\partial x^2}, \qquad Y_2 = \gamma(-\sin nx^3 \tfrac{\partial}{\partial x^1} + \cos nx^3 \tfrac{\partial}{\partial x^2}),$$
$$Y_3 = \mu(\lambda \sin nx^3 \tfrac{\partial}{\partial x^1} - \lambda \cos nx^3 \tfrac{\partial}{\partial x^2} + \tfrac{\partial}{\partial x^3}),$$

where

$$\gamma = \frac{1}{\sqrt{1+\beta}}, \tag{5.18}$$

$$\lambda = \gamma^2 \alpha, \tag{5.19}$$

$$\mu = \frac{2}{n\gamma}. \tag{5.20}$$

For simplicity, we put

$$-\frac{1}{\gamma^2} = a. \tag{5.21}$$

Then we get

$$[Y_1, Y_2] = 0,\ [Y_1, Y_3] = 2aY_2,\ [Y_2, Y_3] = 2Y_1.$$

We have

$$\begin{aligned} &2g(\nabla_{Y_1}Y_2, Y_3) = -2a-2, && 2g(\nabla_{Y_1}Y_3, Y_2) = 2a+2, && 2g(\nabla_{Y_2}Y_1, Y_3) = -2a-2,\\ &2g(\nabla_{Y_2}Y_3, Y_1) = 2a+2, && 2g(\nabla_{Y_3}Y_1, Y_2) = -2a+2, && 2g(\nabla_{Y_3}Y_2, Y_1) = 2a-2, \end{aligned}$$

and the others are equal to 0.

Thus, we have

$$\begin{aligned} &\nabla_{Y_1}Y_1 = 0, && \nabla_{Y_1}Y_2 = -(a+1)Y_3, && \nabla_{Y_1}Y_3 = (a+1)Y_2,\\ &\nabla_{Y_2}Y_1 = -(a+1)Y_3, && \nabla_{Y_2}Y_2 = 0, && \nabla_{Y_2}Y_3 = (a+1)Y_1,\\ &\nabla_{Y_3}Y_1 = -(a-1)Y_2, && \nabla_{Y_3}Y_2 = (a-1)Y_1, && \nabla_{Y_3}Y_3 = 0. \end{aligned}$$

Hence we have

$$\begin{aligned} &R(Y_1, Y_2)Y_1 = -(a+1)^2Y_2, && R(Y_1, Y_2)Y_2 = (a+1)^2Y_1,\\ &R(Y_1, Y_2)Y_3 = 0, && R(Y_1, Y_3)Y_1 = (a+1)(3a-1)Y_3,\\ &R(Y_1, Y_3)Y_2 = 0, && R(Y_1, Y_3)Y_3 = -(a+1)(3a-1)Y_1,\\ &R(Y_2, Y_3)Y_1 = 0, && R(Y_2, Y_3)Y_2 = -(a+1)(a-3)Y_3,\\ &R(Y_2, Y_3)Y_3 = (a+1)(a-3)Y_2. \end{aligned}$$

From the above result, by using (2.8), (5.21) and (5.18), we get

$$\big(Ric(X_i, X_j)\big) = 2\beta \begin{pmatrix} -2-\beta & 0 & 0\\ 0 & 2+\beta & 0\\ 0 & 0 & -\beta \end{pmatrix}. \tag{5.22}$$

Proposition 5.4. *(1) (T^3, g) is η-Einstein if and only if $\beta = 0$ holds.*
(2) (T^3, g) is not Sasakian.
(3) (T^3, g) is not K-contact.

Proof. (1) If (T^3, g) is η-Einstein, then from (2.11) the following equation holds for any $i, j = 1, 2, 3$,

$$Ric(Y_i, Y_j) = pg(Y_i, Y_j) + qg(Y_1, Y_i)g(Y_1, Y_j). \tag{5.23}$$

By substituting $(Y_i, Y_j) = (Y_1, Y_2), (Y_2, Y_2)$ into (5.23), from (5.22) we get

$$Ric(Y_i, Y_j) = 2\beta(2+\beta)g(Y_i, Y_j) - 4\beta(2+\beta)g(Y_1, Y_i)g(Y_1, Y_j). \tag{5.24}$$

Moreover, we substitute $(Y_i, Y_j) = (Y_3, Y_3)$ into (5.24) and get

$$-2\beta^2 = 2\beta(2+\beta).$$

Since (5.13) implies $1+\beta \neq 0$, $\beta = 0$ holds.

Conversely, if $\beta = 0$, (5.24) holds.

(2) If (T^3, g) is Sasakian, then from (2.12) and (2.4) the following equation holds for any $i, j = 1, 2, 3$,

$$R(Y_i, Y_j)Y_1 = g(Y_1, Y_j)Y_i - g(Y_1, Y_i)Y_j. \tag{5.25}$$

By substituting $(Y_i, Y_j) = (Y_1, Y_2), (Y_1, Y_3)$ into (5.25), we get

$$a = 0. \tag{5.26}$$

But since (5.21) implies $a < 0$, (5.26) does not hold. Therefore, (T^3, g) is not Sasakian.

(3) If (T^3, g) is K-contact, then from (2.13) the following equation holds for any $i, k = 1, 2, 3$,

$$2g(Y_k, \nabla_{Y_i} Y_1) + 2g(\nabla_{Y_k} Y_1, Y_i) = 0. \tag{5.27}$$

By substituting $(Y_k, Y_i) = (Y_3, Y_2)$ into (5.27), we get

$$a = 0. \tag{5.28}$$

Similarly, since $a < 0$, (5.28) does not hold. Therefore, (T^3, g) is not K-contact. This completes the proof. □

References

[1] D.E. Blair, Contact manifolds in Riemannian Geometry, Lecture Notes in Mathematics, 509, Springer-Verlag, Berlin, 1976.

[2] D.E. Blair, T. Koufogiorgos and R. Sharma, *A classification of 3-dimensional contact metric manifolds with* $Q\varphi = \varphi Q$, Kodai Math. J., **13**(1990), 391-401.

[3] D.E. Blair and J.N. Patnaik, *Contact manifolds with characteristic vector field annihilated by the curvature*, Bull. Inst. Math. Acad. Sinica, **9**(1981), 533-545.

[4] M. Okumura, *On infinitesimal conformal and projective transformations of normal contact spaces*, Tôhoku Math. J., **14**(1962), 398-412.

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