Dependence of Parameter Estimation on Overlaps

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Motivation

- Which parameter could be well determined?
- Relation between PE and Overlaps
- Studying PN terms of GW template
- Understanding LAL library for KAGALI
- Better overlaps?
CBC GW sources

- Neutron star (NS) + Black hole (BH) binary
- Mass range: $1M_\odot \sim 10^3 M_\odot$
- Frequency range: $1Hz \sim 10kHz$
- Waveform: **Inspiral**, Merger, Ringdown
CBC Inspiral GW Waveform

Inspiral → Merger → Ringdown

Numerical Relativity

Neutron star (NS) and/or black hole (BH)

(PN approximation)

Inspiral
Time Domain Template

- TaylorT1, 2, 3, 4
- PadeT1 : P-approximation
- EOB : effective one body
- SpinTaylorT1, 2, 3, 4, 5
- IMRPhenomA, B, C
Frequency Domain Template

- **TaylorF1**
- **TaylorF2** : standard for detection pipeline, SPA approximation
- **TaylorF2Amp** : include higher harmonics due to amplitude corrections up to 2.5 pN
- **PadeF1**
- **BCV**
- **BCVSpin**
\[ \tilde{h}(f) = \frac{M \nu}{D_L} \sum_{n=0}^{5} \sum_{k=1}^{7} V_k^{2+n} \left( k \frac{dF}{dt} \right)^{-1/2} \times \left( \alpha_k^{(n)} e^{i(2 \pi f t(F) - k \Psi(F) - \pi/4)} + \beta_k^{(n)} e^{i(2 \pi f t(F) - (k \Psi(F) - \pi/2) - \pi/4)} \right), \]

\[ = \frac{M \nu}{D_L} \sum_{n=0}^{5} \sum_{k=1}^{7} V_k^{n-\frac{7}{2}} \sqrt{\frac{5 \pi}{k 48 \nu}} M \left( 1 + S_2 V_k^2 + S_3 V_k^3 + S_4 V_k^4 + S_5 V_k^5 \right) \times \left( \alpha_k^{(n)} + e^{i \pi/2} \beta_k^{(n)} \right) e^{i(k \Psi_{SPA}(f/k) - \pi/4)}, \]

\[ = \frac{M^2}{D_L} \sqrt{\frac{5 \pi \nu}{48}} \sum_{n=0}^{5} \sum_{k=1}^{7} V_k^{n-\frac{7}{2}} C_k^{(n)} e^{i(k \Psi_{SPA}(f/k) - \pi/4)}. \quad (4.72) \]

- Arun et.al., PRD79, 104023(2009)
- Amplitude corrections up to n=5 (2.5pN)
SPA phase factor

\[ \Psi_{\text{SPA}}(F) = 2\pi F t_c - \Psi_c + \frac{3}{256} (2\pi MF)^{-5/3} \left\{ 1 + \left( \frac{3715}{756} + \frac{55}{9} \nu \right) (2\pi MF)^{2/3} + (4\beta - 16\pi) (2\pi MF)^{4/3} \right. \]

\[ + \left( \frac{15293365}{508032} + \frac{27145}{504} \nu + \frac{3085}{72} \nu^2 - 10 \sigma \right) \times \]

\[ (2\pi MF)^{4/3} + \left( \frac{38645}{756} \pi - \frac{65}{9} \pi \nu - \gamma \right) (1 + 3 \log(\nu)) \times (2\pi MF)^{5/3} \left\} \right. \]

(4.81)

- Phase corrections up to 3.5PN (standard in LAL)
int sf2_psi_SPA_coeffs_PN_order(
    REAL8 *PN_coeffs, /* coeef for each PN order*/
    const int phase0,      /* twice PN phase order */
    const sf2_spin_corr_amp *spin_corrections, /* spin correction coeef. Eqs. (4.82,83,84)*/
    const REAL8 eta /* Evans Eq (4.12) */

) {

    REAL8 euler_number = 0.57721566490153286065120900824024310421;
    PN_coeffs[0] = 1.0; /* 0th order newtonian */
    PN_coeffs[1] = 0.0; /* 0.5 order */
    PN_coeffs[2] = 3715.0/756.0 + 55.0*eta/9.0; /* 1.0 order */
    PN_coeffs[3] = 4.0*spin_corrections->beta - 16.0*LAL_PI; /* 1.5 order */
    PN_coeffs[4] = 15293365.0/5080320.0 + 27145.0*eta/504.0 + 3085.0*eta*eta/72.0
                  - 10.0*spin_corrections->sigma; /* 2.0 order */
    PN_coeffs[5] = 36645.0*LAL_PI/756.0 - 65.0*LAL_PI*eta/9.0 - spin_corrections->gamma; /* 2.5 order */
    PN_coeffs[6] = 11583231236531.0/4694215680.0 - 6848.0*euler_number/21.0 - 640.0*LAL_PI*LAL_PI/3.0
                  + (2255.0*LAL_PI*LAL_PI/12.0 - 15737765635.0/3048192.0)*eta
                  + 76055.0*eta*eta/1728.0 - 127825.0*eta*eta*eta/1296.0; /* 3.0 order */
    PN_coeffs[7] = LAL_PI*(77096675.0/254016.0 + 378515.0*eta/1512.0
                        - 74045.0*eta*eta/756.0); /* 3.5 order */
    PN_coeffs[8] = 0.0; /* 4.0 order */
    PN_coeffs[9] = 0.0; /* 4.5 order */
    return 0;
}
Parameter Estimation by MCMC

- **Bayesian Inference**

  \[ \mathcal{P}(\theta|x) = \frac{\mathcal{P}(\theta,x)}{\mathcal{P}(x)} = \frac{\mathcal{P}(x|\theta)\mathcal{P}(\theta)}{\int \mathcal{P}(x|\theta)\mathcal{P}(\theta)d\theta} \]

  - \( \mathcal{P}(\theta, x) = \mathcal{P}(x|\theta)\mathcal{P}(\theta) = \mathcal{P}(\theta|x)\mathcal{P}(x) \)
  
  - \( \theta \): unobservable model parameters
  
  - \( x \): observable data
  
  - \( \mathcal{P}(\theta, x) \): Joint probability observing data \( x \) with model parameter \( \theta \)
  
  - \( \mathcal{P}(x|\theta) \propto \text{Likelihood function} \)

- **Metropolis-Hasting Algorithm**, \( 10^6 \sim 10^7 \) samples
Overlap Calculation

\[ \rho = 4 \Re \left( \int_{0}^{\infty} \frac{\tilde{h}(f) \hat{s}^*(f)}{S_n(f)} \, df \right) \]

One-sided power spectral density of noise

\[ SNR^2 = 4 \int_{0}^{\infty} \frac{|\tilde{h}(f)|^2}{S_n(f)} \, df \]

Network SNR

\[ SNR^2 = \sum_{i=1}^{N} SNR_i^2 \]
Likelihood Calculation

\[ \mathcal{L}(s \mid \theta) \propto \exp \left[ -2 \int_0^\infty \frac{|\tilde{s}(f) - \tilde{h}(\theta, f)|^2}{S_n(f)} df \right] \]
Various Overlaps

- **Unnormalized**

  \[ \rho = 4 \Re \left( \int_0^{+\infty} \frac{\tilde{h}_1(f)\tilde{h}_2^*(f)}{S_n(f)} df \right) \]

- **Normalized**

  \[ \rho_k = 4 \Re \left( \int_0^{+\infty} \frac{\tilde{h}_k(f)\tilde{h}_k^*(f)}{S_n(f)} df \right) \]

  \[ \rho_N = \frac{\rho}{\sqrt{\rho_1 \rho_2}} \]
Sensitivities of 2nd Gen. Detectors


Virgo: [http://wwwcascina.virgo.infin.it/advirgo](http://wwwcascina.virgo.infin.it/advirgo)

3.0 \times 10^{-24}/\sqrt{Hz}
Sensitivities of 2\textsuperscript{nd} Gen. Detectors
Sensitivities as configurations

Ref. : LIGO-T0900288-v3
## Injection Values for BH-NS

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_1 [M_\odot]$</td>
<td>10.0</td>
<td>Polarization [rad]</td>
<td>2.606</td>
</tr>
<tr>
<td>$M_1 [M_\odot]$</td>
<td>1.4</td>
<td>Orbital phase [rad]</td>
<td>3.31</td>
</tr>
<tr>
<td>$M_{\text{chirp}} [M_\odot]$</td>
<td>2.9943</td>
<td>Coalescence Time</td>
<td>894383679</td>
</tr>
<tr>
<td>Sym. Mass ratio</td>
<td>0.1077</td>
<td>Ra [rad]</td>
<td>0.645</td>
</tr>
<tr>
<td>Dist [Mpc]</td>
<td>325</td>
<td>Dec [rad]</td>
<td>0.575</td>
</tr>
<tr>
<td>Inclination [rad]</td>
<td>$\pi/4$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Non-spinning BH-NS binary with aLIGO-aVirgo 3-detector network sensitivity with S/N ~ 20
Posterior SD and Overlap FWHM
Posterior SD and Overlap FWHM
Posterior SD and Overlap FWHM

Probability Density

$m_1 (M_\odot)$
Posterior SD and Overlap FWHM
Posterior SD and Overlap FWHM

![Histogram of $m_2$ (M$_\odot$) probability density](image-url)
Posterior SD and Overlap FWHM

Overlaps TaylorF2Amp-TaylorF2Amp w.r.t. Parameters

Normalized Overlap

Overlap

m2

Normalised
Unnormalised
Likelihood
Diff Unnormalised
Posterior SD and Overlap FWHM
Posterior SD and Overlap FWHM

Overlaps TaylorF2Amp-TaylorF2Amp w.r.t. Parameters

Normalized Overlap

Overlap

ra

Normalised
Unnormalised
Likelihood
Diff Unnormalised

2014-12-20
7th Korea-Japan Workshop on KAGRA
Posterior SD and Overlap FWHM
Posterior SD and Overlap FWHM
## SDs for BH(10)-NS(1.4)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Standard Deviation</th>
<th>Relative(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dist[Mpc]</td>
<td>53.58</td>
<td>16.5</td>
</tr>
<tr>
<td>M1[M☉]</td>
<td>$6.954 \times 10^{-2}$</td>
<td>0.70</td>
</tr>
<tr>
<td>M2[M☉]</td>
<td>$8.248 \times 10^{-3}$</td>
<td>0.59</td>
</tr>
<tr>
<td>Ra[rad]</td>
<td>$1.144 \times 10^{-2}$</td>
<td>0.18(Note 1)</td>
</tr>
<tr>
<td>Dec[rad]</td>
<td>$1.913 \times 10^{-2}$</td>
<td>1.22(Note 2)</td>
</tr>
</tbody>
</table>

Note 1: relative error with respect to $2\pi$
Note 2: relative error with respect to $\pi/2$
## FWHMs for BH(10)-NS(1.4)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Overlap</th>
<th>Likelihood</th>
<th>Relative(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dist[Mpc]</td>
<td>$\infty$</td>
<td>24.5</td>
<td>7.5</td>
</tr>
<tr>
<td>M1[$M_\odot$]</td>
<td>0.021</td>
<td>$&lt; 0.001$</td>
<td>$&lt; 0.01$</td>
</tr>
<tr>
<td>M2[$M_\odot$]</td>
<td>0.0046</td>
<td>$&lt; 0.0003$</td>
<td>$&lt; 0.02$</td>
</tr>
<tr>
<td>Ra[rad]</td>
<td>0.225</td>
<td>0.081</td>
<td>1.3$^{(Note 1)}$</td>
</tr>
<tr>
<td>Dec[rad]</td>
<td>1.695</td>
<td>0.255</td>
<td>8.1$^{(Note 2)}$</td>
</tr>
</tbody>
</table>

Note 1: relative error with respect to $2\pi$

Note 2: relative error with respect to $\pi/2$
## PE Dependence on Overlaps

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Overlap</th>
<th>SD</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dist[MPc]</td>
<td>∞</td>
<td>16.5</td>
<td>0</td>
</tr>
<tr>
<td>M1[M☉]</td>
<td>0.21</td>
<td>0.70</td>
<td>3.33</td>
</tr>
<tr>
<td>M2[M☉]</td>
<td>0.33</td>
<td>0.59</td>
<td>1.79</td>
</tr>
<tr>
<td>Ra[rad]</td>
<td>3.6</td>
<td>0.18</td>
<td>0.05</td>
</tr>
<tr>
<td>Dec[rad]</td>
<td>54.0</td>
<td>1.22</td>
<td>0.023</td>
</tr>
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</table>
## PE Dependence on Overlaps

<table>
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<th>Parameter</th>
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<th>SD</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dist[Mpc]</td>
<td>7.5</td>
<td>16.5</td>
<td>2.2</td>
</tr>
<tr>
<td>M1[M⨀]</td>
<td>0.01</td>
<td>0.7</td>
<td>70.0</td>
</tr>
<tr>
<td>M2[M⨀]</td>
<td>0.02</td>
<td>0.59</td>
<td>29.5</td>
</tr>
<tr>
<td>Ra[rad]</td>
<td>1.3</td>
<td>0.18</td>
<td>0.14</td>
</tr>
<tr>
<td>Dec[rad]</td>
<td>8.1</td>
<td>1.22</td>
<td>0.15</td>
</tr>
</tbody>
</table>
PE Dependence on Overlaps

![Graph showing PE dependence on Overlaps]
PE Dependence on Overlaps

PE dependence on Overlaps

Relative Error in Overlap/Likelihood [%] vs. Relative Error in 1D Posterior [%]

- m1
- m2
- ra
- dec
- dist
Remarks

- 1D Posterior is marginalized, overlap is not
- Estimation clue from overlap
- The sharper peak in likelihood is the smaller SD in 1D posterior
- Better measure resolving degeneracy?