Möbius numbers of some modified generalized noncrossing partitions

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Dedicated to Professor Jun Morita on the occasion of his 60th birthday

Abstract.
In this paper we will compute the Möbius number of \( \{ \text{NC}^{(k)}(W) \setminus \text{mins} \} \cup \{ \emptyset \} \) for a Coxeter group \( W \) which contains an affirmative answer to conjecture 3.7.9 in [1].

1. Introduction

In this paper we will prove the following theorem which yields an affirmative answer to a conjecture by Armstrong [[1], conjecture 3.7.9].

Theorem 1.1. For each finite Coxeter group \((W, S)\) with \(|S| = n\) and for all positive integers \(k\), the Möbius number of \( \{ \text{NC}^{(k)}(W) \setminus \text{mins} \} \cup \{ \emptyset \} \) is equal to \((-1)^n \left( \text{Cat}^{(k)}_+(W) - \text{Cat}^{(k-1)}_+(W) \right)\).

In [2] Armstrong and Krattenthaler proved this result by counting the multichains of \( \text{NC}^{(k)}(W) \). Moreover they proved this result for the case of well-generated complex reflection groups. Our approach is easier and different from theirs. Our method is using the EL-labeling of \( \text{NC}^{(k)}(W) \) introduced by Armstrong and Thomas [1].

If one has an EL-labeling for \( \text{NC}(W) \) for any complex reflection group \( W \), then one can state our Theorem 1.1 in the case of any well-generated

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complex reflection group from the method introduced by Armstrong and Thomas at Section 3.7 in [1]. Athanasiadis, Brady and Watt gave an EL-labeling for NC(W) using some properties of the root system derived from a real reflection group W [3]. Recently M"uhle proved that the poset NC(W) is an EL-shellable poset for any well-generated complex reflection group W that is not a Coxeter group [6].

Remark 1.1. The result in this paper was obtained when the author was Jun Morita’s graduate student at University of Tsukuba. The result was submitted to the preprint server arXiv as arXiv:0905.1660 at 11th May, 2009. Independently, Armstrong and Krattenthaler obtained the result for the case of well-generated complex reflection groups and they also submitted to arXiv as arXiv:0905.0205 at 2nd May, 2009. Henri M"uhle proved that the poset NC(W) is an EL-shellable poset and he submitted to arXiv as arXiv:1111.7172 at 30th November, 2011.

2. Preliminaries

2.1. Generalized noncrossing partitions and Fuss–Catalan numbers

Let (W, S) be a Coxeter system with |S| = n and |W| < \infty. Basic properties of Coxeter groups are introduced in [5]. We set $T := \{sws^{-1} \mid s \in S, w \in W\}$ which is the conjugate closure of the generating set S and let $l_{T} : W \to \mathbb{Z}$ denote the word length on W with respect to the set T. The function $l_{T}$ naturally induces a partial order on W by setting $\pi \leq_{T} \sigma$ if $l_{T}(\sigma) = l_{T}(\pi) + l_{T}(\pi^{-1}\sigma)$ and we call it the absolute order on W. Fix a Coxeter element $\gamma \in W$ and set $\text{NC}(W) := [e, \gamma]$. The reader finds that the poset NC(W) is well-defined because Coxeter elements form a conjugacy class and hence $[e, \gamma_1] \simeq [e, \gamma_2]$ for Coxeter elements $\gamma_1, \gamma_2$.

Next we set $\text{NC}^{(k)}(W) := \{ (\pi_1, \ldots, \pi_k) \mid \pi_i \in \text{NC}(W) \text{ for } 1 \leq i \leq k \text{ with } \pi_1 \leq \pi_2 \leq \cdots \leq \pi_k \leq \gamma \}$ and $\text{NC}'^{(k)}(W) := \{ (\delta_1, \ldots, \delta_k) \mid \delta_i \in \text{NC}(W) \text{ for } 1 \leq i \leq k \text{ with } l(\delta_1 \cdots \delta_i) = l(\delta_1) + \cdots + l(\delta_i) \text{ for } 1 \leq i \leq k \}$. In Section 3.3 of [1], Armstrong introduced the order structure for $\text{NC}^{(k)}(W)$ as follows:

for
\( (\pi_k^{(1)}) := (\pi_1^{(1)}, \ldots, \pi_k^{(1)}) \),
\( (\pi_k^{(2)}) := (\pi_1^{(2)}, \ldots, \pi_k^{(2)}) \in \text{NC}^{(k)}(W) \),
\( (\pi_k^{(1)}) \leq (\pi_k^{(2)}) \iffdef \frac{(\pi_1^{(2)})_{i+1}}{(\pi_1^{(1)})_{i+1}} \leq \frac{(\pi_1^{(1)})_i}{(\pi_1^{(2)})_i} \) in \( \text{NC}(W) \)
for \( 1 \leq i \leq k \) with \( \pi_{k+1}^{(1)} = \pi_{k+1}^{(2)} = \gamma \), and he also defined for \( \text{NC}(k)(W) \) as follows:
for
\( (\delta_k^{(1)}) := (\delta_1^{(1)}, \ldots, \delta_k^{(1)}) \),
\( (\delta_k^{(2)}) := (\delta_1^{(2)}, \ldots, \delta_k^{(2)}) \in \text{NC}(k)(W) \),
\( (\delta_k^{(1)}) \leq (\delta_k^{(2)}) \iffdef \delta_i^{(1)} \leq \delta_i^{(2)} \) in \( \text{NC}(W) \) for \( 1 \leq i \leq k \).

The reader finds that the poset \( \text{NC}(k)(W) \) is the dual poset of \( \text{NC}^{(k)}(W) \) (for more information, see [1]).

We can define Fuss-Catalan numbers and positive Fuss-Catalan numbers for finite Coxeter groups [1].

**Definition 2.1** ([1]). Let \( (W, S) \) be a finite Coxeter system of rank \( |S| \) and let \( d_1, d_2, \ldots, d_n \) be its degrees. We define

1. \( \text{Cat}^{(k)}(W) := \prod_{i=1}^{n} \frac{kh+d_i}{d_i} = \frac{1}{|W|} \prod_{i=1}^{n} (kh+d_i) \), to be the Fuss–Catalan number, see Definition 3.5.1 of [1],
2. \( \text{Cat}^{(k)}_+(W) := \prod_{i=1}^{n} \frac{kh+d_i-2}{d_i} = \frac{1}{|W|} \prod_{i=1}^{n} (kh+d_i-2) \), to be the positive Fuss–Catalan number, see Definition 3.7.5 of [1],

where \( k \in \mathbb{N} \) and \( h \) is the Coxeter number of \( W \).

The number of the elements of the generalized noncrossing partition \( \text{NC}^{(k)}(W) \) is enumerated by the Fuss–Catalan number corresponding to \( k \) and \( W \), see Theorem 3.5.3 in [1].
2.2. EL-shellability

Let \((P, \preceq)\) be a finite poset. We say that a poset \(P\) is bounded if it has a maximum element \(\hat{1}\) and a minimum element \(\hat{0}\). Also \(P\) is called graded if all maximal chains in \(P\) have the same length and the length is denoted by \(\text{rank}(P)\). If \(P\) is a graded and bounded poset, let \(\text{rank}(x)\) denote the length of an unrefinable maximal chain of the poset \([\hat{0}, x]\) for \(x \in P\). Let \(\epsilon(P)\) be the set of covering relations of \(P\), meaning pairs \((x, y)\) of elements of \(P\) such that \(y\) covers \(x\), we denote it by \(x \prec y\), in \(P\). Let \(\Lambda\) be a totally ordered set. An edge labeling of \(P\) with the label set \(\Lambda\) is a map \(\lambda : \epsilon(P) \rightarrow \Lambda\). Let \(c\) be an unrefinable chain \(x_0 \prec x_1 \prec \cdots \prec x_r\) of elements of \(P\) so that \((x_{i-1}, x_i) \in \epsilon(P)\) for all \(1 \leq i \leq r\). We let \(\lambda(c) = (\lambda(x_0, x_1), \lambda(x_1, x_2), \cdots, \lambda(x_{r-1}, x_r))\) be the label of \(c\) with respect to \(\lambda\) and call \(c\) a rising chain and a falling chain with respect to \(\lambda\) if the entries of \(\lambda(c)\) strictly increase or weakly decrease, respectively, in the total order of \(\Lambda\). We say that \(c\) is lexicographically smaller than an unrefinable chain \(c'\) in \(P\) with respect to \(\lambda\) if \(\lambda(c)\) precedes \(\lambda(c')\) in the lexicographic order induced by the total order of \(\Lambda\) [3].

**Definition 2.2** ([4]). An edge labeling \(\lambda\) of \(P\) is called an EL-labeling if for every nonsingleton interval \([u, v]\) in \(P\)

1. there is a unique rising maximal chain in \([u, v]\) and
2. this chain is lexicographically smallest among all maximal chains in \([u, v]\) with respect to \(\lambda\).

The poset \(P\) is called EL-shellable if it has EL-labeling for some label set \(\Lambda\). For a graded and bounded poset \((P, \preceq)\), we denote by \(\mu(P)\) the Möbius number of \(P\). By using the EL-labeling we can compute the Möbius number of \(P\).

**Theorem 2.1** ([7]). If \(P\) is EL-shellable then the Möbius number of \(P\) is the number of falling maximal chains of \(P\) up to sign \((-1)^{\text{rank}(P)}\).

3. Main result

In this section we prove Theorem 1.1.
For $k \in \mathbb{N}$ and an arbitrary finite Coxeter group $(W, S)$, we consider the poset $NC(k)(W)$ which is the dual poset of $NC(k)(W)$. We put $\text{maxs}$ to be the set of maximal elements of $NC(k)(W)$. The poset $\{NC(k)(W) \setminus \text{maxs}\} \cup \{\hat{1}\}$ is the dual of $\{NC(k)(W) \setminus \text{mins}\} \cup \{\hat{0}\}$ hence it is sufficient to prove $\mu(\{NC(k)(W) \setminus \text{maxs}\} \cup \{\hat{1}\}) = (-1)^n \left(\text{Cat}_+(k)(W) - \text{Cat}^{(k-1)}(W)\right)$ to show our Theorem 1.1. It is easy to see the following Lemma.

**Lemma 3.1.** Let $P$ be a graded poset with a minimum element $\hat{0}$. We put $\text{maxs}(P)$ the set of maximal elements of $P$. Then the poset $P \setminus \text{maxs}(P)$ is also graded. We denote by $\mu(P \setminus \text{maxs}(P))$ the Möbius number of $\{P \setminus \text{maxs}(P)\} \cup \{\hat{1}\}$. Then we have $\mu(P \setminus \text{maxs}(P)) \cup \{\hat{1}\} = \mu(P \cup \{\hat{1}\}) + \sum_{x \in \text{maxs}(P)} \mu([0, x])$.

In [1] Armstrong and Thomas gave an EL-labeling of $NC(k)(W) \cup \{\hat{1}\}$. Recall that the edges in the Hasse diagram of $NC(W)$ are naturally labeled by reflections $T = \{wsw^{-1} \mid s \in S, w \in W\}$. Athanasiadis, Brady and Watt defined a total order on the set $T$ such that the natural edge-labeling by $T$ becomes an EL-labeling of the poset $NC(W)$. We denote the EL-labeling by $\lambda : (NC(W)) \rightarrow T$. In [3] they called the total order on $T$ the **ABW order**. They put $T := \{t_1, \cdots, t_N\}$ with the **ABW order** $t_1 < t_2 < \cdots < t_N$. Recall that $NC(W)^{(k)}$ is edge-labeled by the set of $i$-th reflections $T^k := \{t_{i,j} = (1, 1, \cdots, t_j, \cdots, 1) : 1 \leq i, j \leq N\}$ where $t_j$ occurs in the $i$-th entry of $t_{i,j}$. Armstrong and Thomas defined the **lex ABW order** on $T^k$ as $t_{1,1} < t_{1,2} < \cdots < t_{1,N} < t_{2,1} < t_{2,2} < \cdots < t_{2,N} < \cdots < t_{k,1} < t_{k,2} < \cdots < t_{k,N}$. This induces an EL-shelling of $NC(W)^k$. Now recall that $NC(k)(W)$ is an order ideal in $NC(W)^k$, so the **lex ABW order** on $T^k$ induces an EL-labeling of the Hasse diagram of $NC(k)(W)$. They considered the set $T^k \cup \{\theta\}$ with $t_{1,1} < t_{1,2} < \cdots < t_{1,N} < \theta < t_{2,1} < t_{2,2} < \cdots < t_{2,N} < \cdots < t_{k,1} < t_{k,2} < \cdots < t_{k,N}$. For $x \in \text{maxs}$, they put $\lambda(x, \hat{1}) := \theta$, where $(x, \hat{1})$ is the edge from $x$ to $\hat{1}$. They showed that the labeling induces an EL-labeling of $NC(k)(W) \cup \{\hat{1}\}$. Now we denote their EL-labeling by $\hat{\lambda} : \epsilon(\text{NC}(k)(W) \cup \{\hat{1}\}) \rightarrow T^k \cup \{\theta\}$.

We have
\[
\begin{align*}
\mu\left(\{\text{NC}^{(k)}(W) \setminus \text{mins}\} \cup \{\hat{0}\}\right) &= \mu\left(\{\text{NC}^{(k)}(W) \setminus \text{maxs}\} \cup \{\hat{1}\}\right) \\
&= \sum_{x \in \text{maxs}} \mu(\hat{0}, x) + \mu(\text{NC}^{(k)}(W) \cup \{\hat{1}\}) \\
&= \sum_{x \in \text{maxs}} \mu(\hat{0}, x) + (-1)^{n-1}\text{Cat}_{+}^{(k-1)}(W)
\end{align*}
\]

because \(\mu(\text{NC}^{(k)}(W) \cup \{\hat{1}\}) = (-1)^{n-1}\text{Cat}_{+}^{(k-1)}(W)\) from Theorem 3.7.7 in [1].

**Proposition 3.2.** Notation is as above, then we have

\[
\sum_{x \in \text{maxs}} \mu(\hat{0}, x) = (-1)^n \text{Cat}^{(k)}_{+}(W)
\]

where \(\text{maxs} := \{(\delta_1, \cdots, \delta_k)|l(\delta_1 \cdots \delta_i) = l(\delta_1) + \cdots + l(\delta_i) \text{ for } 1 \leq i \leq k \text{ with } \delta_1 \cdots \delta_k = c\}\).

**Proof.**

For \((\delta_1, \cdots, \delta_k) \in \text{maxs}\), the reader finds that \([(e, \cdots, e), (\delta_1, \cdots, \delta_k)] \simeq [e, \delta_1] \times [e, \delta_2] \times \cdots [e, \delta_k]\) and hence we obtain

\[
\sum_{x \in \text{maxs}} \mu(\hat{0}, x) = \sum_{(\delta_1, \cdots, \delta_k), l(\delta_1 \cdots \delta_i) = l(\delta_1) + \cdots + l(\delta_i) \text{ for } 1 \leq i \leq k \text{ with } \delta_1 \cdots \delta_k = c} \mu([e, \delta_1]) \cdots \mu([e, \delta_k]).
\]

To show \(\sum_{(\delta_1, \cdots, \delta_k), l(\delta_1 \cdots \delta_i) = l(\delta_1) + \cdots + l(\delta_i) \text{ for } 1 \leq i \leq k \text{ with } \delta_1 \cdots \delta_k = c} \mu([e, \delta_1]) \cdots \mu([e, \delta_k]) = (-1)^n\text{Cat}^{(k)}_{+}(W)\), we consider the EL-labeling of \(\text{NC}^{(k+1)}(W) \cup \{\hat{1}\}\), not \(\text{NC}^{(k)}(W) \cup \{\hat{1}\}\), introduced by Armstrong and Thomas. Recall that \(\mu(\text{NC}^{(k+1)}(W) \cup \{\hat{1}\})\) equals the number of the falling maximal chains of \(\text{NC}^{(k+1)}(W) \cup \{\hat{1}\}\) with respect to \(\hat{\lambda}\) up to sign \((-1)^n\).

Let \(c\) be an unrefinable chain \((e, \cdots, e) \prec \cdots \prec (\delta_1, \cdots, \delta_{k+1}) \prec \hat{1}\) of elements of \(\text{NC}^{(k+1)}(W) \cup \{\hat{1}\}\). If \(c\) is a falling maximal chain with respect to \(\hat{\lambda}\), we must have \(\delta_1 = e\) because \(\hat{\lambda}(\delta_1, \cdots, \delta_{k+1}, \hat{1})\) equals to \(\theta\) and the number \(\theta\) is bigger than \(t_{1,i}\) for \(1 \leq i \leq N\) in the total order on \(T^{k+1} \cup \{\theta\}\).

Moreover

\(c\) is a falling maximal chain
if and only if

\[
\begin{align*}
\text{each of the chains } & \quad \text{length}=l(\delta_{k+1}) \\
(e, \cdots, e) \prec \cdots \prec (e, \cdots, e, \delta_{k+1}), & \quad \text{length}=l(\delta_k), \\
(e, \cdots, e, \delta_1, \cdots, \delta_{k+1}) \prec \cdots \prec (e, \cdots, e, \delta_1, \delta_{k+1}), & \quad \text{length}=l(\delta_{i_1}), \\
(e, \delta_3, \cdots \delta_{k+1}) \prec \cdots \prec (e, \delta_2, \delta_3, \cdots \delta_{k+1}) & \quad \text{is a falling unrefinable chain in } NC_{(k+1)}(W) \cup \{\tilde{1}\}.
\end{align*}
\]

Now we denote the number of the falling maximal chains from \(e\) to \(\delta \in NC(W)\) with respect to \(\lambda\) by \(CH(NC(W), \delta, \lambda)\). Then we have

\[
\begin{align*}
\mu(NC_{(k+1)}(W) \cup \{\tilde{1}\}) &= (-1)^n CH(NC_{(k+1)}(W) \cup \{\tilde{1}\}, \tilde{1}, \lambda) \\
&= \sum_{(e, \delta_2, \cdots, \delta_{k+1}) \in \text{maxs}} (-1)^n CH(NC(W), \delta_2, \lambda) \cdots CH(NC(W), \delta_{k+1}, \lambda) \\
&= \sum_{(e, \delta_2, \cdots, \delta_{k+1}) \in \text{maxs}} (-1)^{l(\delta_2)} CH(NC(W), \delta_2, \lambda) \cdots (-1)^{l(\delta_{k+1})} CH(NC(W), \delta_{k+1}, \lambda) \cdot (-1) \\
&= \sum_{(e, \delta_2, \cdots, \delta_{k+1}) \in \text{maxs}} \mu([e, \delta_2]) \cdots \mu([e, \delta_{k+1}]) \cdot (-1) \\
&= \sum_{(\delta_1, \delta_2, \cdots, \delta_k) : \delta_1 \delta_2 \cdots \delta_k = e, l(\delta_1) + l(\delta_2) + \cdots + l(\delta_k) = n-1} \mu([e, \delta_1]) \cdots \mu([e, \delta_k]) \cdot (-1) \\
&= \sum_{x \in \text{maxs}} \mu(\tilde{0}, x) \cdot (-1).
\end{align*}
\]

Also \(\mu(NC_{(k+1)}(W) \cup \{\tilde{1}\}) = (-1)^{n-1} \text{Cat}_{+}^{(k)}(W)\) [1] and hence we obtain \(\sum_{x \in \text{maxs}} \mu(\tilde{0}, x) = (-1)^n \text{Cat}_{+}^{(k)}(W)\). This completes the proof.

The proof of Theorem 1.1 follows from the previous arguments.

4. Henri Mühle’s generalization

Recently, it was shown that the poset \(NC(W)\) is EL-shellable for any well-generated complex reflection group \(W\) in Theorem 1.3 of [6], and con-
sequently we can adapt our proof of Theorem 1.1 to show the following, more general statement, which has appeared in Corollary 6.2 in [6].

**Theorem 4.1** ([6]). *For a well-generated complex reflection group of rank $n$ and a positive integer $k$, the Möbius number of $\{\text{NC}^{(k)}(W) \setminus \text{mins} \} \cup \{0\}$ is equal to $(-1)^n \left( \text{Cat}^{(k)}_{+}(W) - \text{Cat}^{(k-1)}_{+}(W) \right)$.*

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