Root graded Lie superalgebras which appear as the centerless cores of extended affine Lie superalgebras

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Dedicated to Professor Jun Morita on the occasion of his 60th birthday.

Abstract. We recall the notions of extended affine Lie superalgebras and root graded Lie superalgebras. We show how a root graded Lie superalgebra can appear as the centerless core of an extended affine Lie superalgebra.

1. Extended Affine Lie superalgebras

Throughout this work, \( \mathbb{F} \) is a field of characteristic zero. Unless otherwise mentioned, all vector spaces are considered over \( \mathbb{F} \). We denote the dual space of a vector space \( V \) by \( V^* \). If \( V \) is a vector space graded by an abelian group, we denote the degree of a homogeneous element \( x \in V \) by \( |x| \); we also make a convention that if \( |x| \) is appeared in an expression, for an element \( x \) of \( V \), by default, we assume that \( x \) is homogeneous. If \( X \) is a subset of a group \( A \), the symbol \( \langle X \rangle \) means the subgroup of \( A \) generated by \( X \). Also we denote the cardinal number of a set \( S \) by \( |S| \). We set \( \delta_{i,j} := 0 \) if \( i \neq j \) and \( \delta_{i,j} := 1 \) if \( i = j \), the Kronecker delta. For a map \( f : A \to B \) and \( C \subseteq A \), by \( f \restriction_C \), we mean the restriction of \( f \) to \( C \). Also we use \( \cup \) to indicate the disjoint

2000 Mathematics Subject Classification. 17B67.

Key words and phrases. Extended affine Lie superalgebras, Root graded Lie superalgebras.

*This research was in part supported by a grant from IPM (No. 92170415) and partially carried out in IPM-Isfahan branch.
union.

In the present paper, by a symmetric form on an additive abelian group $A$, we mean a map $(\cdot, \cdot) : A \times A \to F$ satisfying

- $(a, b) = (b, a)$ for all $a, b \in A$,
- $(a+b, c) = (a, c)+(b, c)$ and $(a, b+c) = (a, b)+(a, c)$ for all $a, b, c \in A$.

In this case, we set $A^0 := \{a \in A \mid (a, A) = \{0\}\}$ and call it the radical of the form $(\cdot, \cdot)$. The form is called nondegenerate if $A^0 = \{0\}$. If $A$ is a vector space over $F$, bilinear forms are used in the usual sense.

We call a triple $(L, H, (\cdot, \cdot))$, consisting of a nonzero Lie superalgebra $L = L_0 \oplus L_1$, a nontrivial subalgebra $H$ of $L_0$ and a nondegenerate invariant even supersymmetric bilinear form $(\cdot, \cdot)$ on $L$, a super-toral triple if

- $L$ has a weight space decomposition $L = \oplus_{\alpha \in H^*} L^\alpha$ with respect to $H$ via the adjoint representation. We note that in this case $H$ is abelian; also as $L_0$ as well as $L_1$ are $H$-submodules of $L$, we have $L_0 = \oplus_{\alpha \in H} L_0^\alpha$ and $L_1 = \oplus_{\alpha \in H} L_1^\alpha$ with $L_i^\alpha := L_i \cap L^\alpha$, $i = 0, 1$ [4, Pro. 2.1.1],

- the restriction of the form $(\cdot, \cdot)$ to $H$ is nondegenerate.

We call $R := \{\alpha \in H^* \mid L^\alpha \neq \{0\}\}$, the root system of $L$ (with respect to $H$). Each element of $R$ is called a root. We refer to elements of $R_0 := \{\alpha \in H^* \mid L_0^\alpha \neq \{0\}\}$ (resp. $R_1 := \{\alpha \in H^* \mid L_1^\alpha \neq \{0\}\}$) as even roots (resp. odd roots). We note that $R = R_0 \cup R_1$. Suppose that $(L, H, (\cdot, \cdot))$ is a super-toral triple and $\eta : H \to H^*$ is the function mapping $h \in H$ to $(h, \cdot)$. Since the form is nondegenerate on $H$, this map is one to one. So for each element $\alpha$ of the image $H^\eta$ of $H$ under $\eta$, there is a unique $t_\alpha \in H$ representing $\alpha$ through the form $(\cdot, \cdot)$. Now we can transfer the form on $H$ to a form on $H^\eta$, denoted again by $(\cdot, \cdot)$, and defined by

$$(\alpha, \beta) := (t_\alpha, t_\beta) \quad (\alpha, \beta \in H^\eta).$$

Lemma 1.1 ([9, Lem. 3.1]). Suppose that $(L, H, (\cdot, \cdot))$ is a super-toral triple with corresponding root system $R = R_0 \cup R_1$. Then we have the following:
(i) For $\alpha, \beta \in H$, $[L_\alpha, L_\beta] \subseteq L_{\alpha + \beta}$. Also for $i = 0, 1$ and $\alpha, \beta \in R_i$, we have $(L_\alpha^i, L_\beta^i) = \{0\}$ unless $\alpha + \beta = 0$; in particular, $R_0 = -R_0$ and $R_1 = -R_1$.

(ii) Suppose that $\alpha \in H^p$ and $x_\pm \alpha \in L_{\pm \alpha}$ with $[x_\alpha, x_{-\alpha}] \in H$, then we have $[x_\alpha, x_{-\alpha}] = (x_\alpha, x_{-\alpha})_{\alpha}$.

(iii) Suppose that $\alpha \in R_i \setminus \{0\}$ (i.e. $\alpha \in \mathbb{H} \setminus \{0\}$), $x_\alpha \in L_{\alpha}^i$ and $x_{-\alpha} \in L_{-\alpha}^i$ with $[x_\alpha, x_{-\alpha}] \in H \setminus \{0\}$, then we have $(x_\alpha, x_{-\alpha}) \neq 0$ and that $\alpha \in H^p$.

Definition 1.1. A super-toral triple $(L = L_0 \oplus L_1, H, (\cdot, \cdot))$ (or $L$ if there is no confusion), with root system $R = R_0 \cup R_1$, is called an extended affine Lie superalgebra if

- (1) for $\alpha \in R_i \setminus \{0\}$ (i.e. $\alpha \in \mathbb{H} \setminus \{0\}$), there are $x_\alpha \in L_\alpha^i$ and $x_{-\alpha} \in L_{-\alpha}^i$ such that $0 \neq [x_\alpha, x_{-\alpha}] \in H$,

- (2) for $\alpha \in R$ with $(\alpha, \alpha) \neq 0$ and $x \in L_\alpha$, $\text{ad}_x : L \to L$, mapping $y \in L$ to $[x, y]$, is a locally nilpotent linear transformation.

For an extended affine Lie superalgebra $(L, H, (\cdot, \cdot))$ with root system $R$, the subsuperalgebra of $L$ generated by $L_\alpha$, for $\alpha \in R$, is called the core of $L$. $(L, H, (\cdot, \cdot))$ is called an invariant affine reflection algebra if $L_1 \neq \{0\}$ and it is called a locally extended affine Lie algebra if $L_1 \neq \{0\}$ and $L_0 = H$. Finally a locally extended affine Lie algebra $(L, H, (\cdot, \cdot))$ is called an extended affine Lie algebra if $L_0 = H$ is a finite dimension subalgebra of $L$.

We immediately have the following proposition:

Proposition 1.2. If $(L, H, (\cdot, \cdot))$ is an extended affine Lie superalgebra, then the triple $(L_0, H, (\cdot, \cdot)|_{L_0 \times L_0})$ is an invariant affine reflection algebra.


One knows from [9, Cor. 3.9] that the root system of an extended affine Lie superalgebra $(L, H, (\cdot, \cdot))$ is an extended affine root supersystem in the following sense:
Definition 1.2. Suppose that $A$ is a nontrivial additive abelian group, $(\cdot, \cdot) : A \times A \to \mathbb{F}$ is a symmetric form and $R$ is a subset of $A$. Set

$$R^0 := R \cap A^0, \quad R^\times := R \setminus R^0,$$

$$R_{re}^\times := \{ \alpha \in R \mid (\alpha, \alpha) \neq 0 \}, \quad R_{re} := R_{re}^\times \cup \{0\},$$

$$R_{ns}^\times := \{ \alpha \in R \setminus R^0 \mid (\alpha, \alpha) = 0 \}, \quad R_{ns} := R_{ns}^\times \cup \{0\}.$$

We say $(A, (\cdot, \cdot), R)$ is an extended affine root supersystem if the following hold:

1. $(S1)$ $0 \in R$, and $\langle R \rangle = A$,
2. $(S2)$ $R = -R$,
3. $(S3)$ for $\alpha \in R_{re}^\times$ and $\beta \in R$, $2(\alpha, \beta)/(\alpha, \alpha) \in \mathbb{Z}$,
   
   root string property holds in $R$ in the sense that for $\alpha \in R_{re}^\times$ and $\beta \in R$, there are nonnegative integers $p, q$ with $2(\beta, \alpha)/(\alpha, \alpha) = p - q$ such that

   $$\{ \beta + k\alpha \mid k \in \mathbb{Z} \} \cap R = \{ \beta - p\alpha, \ldots, \beta + q\alpha \},$$

4. $(S4)$ $\{ \beta + k\alpha \mid k \in \mathbb{Z} \} \cap R = \{ \beta - p\alpha, \ldots, \beta + q\alpha \}$,
5. $(S5)$ for $\alpha \in R_{ns}$ and $\beta \in R$ with $(\alpha, \beta) \neq 0$, $\{ \beta - \alpha, \beta + \alpha \} \cap R \neq \emptyset$.

If there is no confusion, for the sake of simplicity, we say $R$ is an extended affine root supersystem in $A$. Elements of $R^0$ are called isotropic roots, elements of $R_{re}$ are called real roots and elements of $R_{ns}$ are called nonsingular roots. A subset $X$ of $R^\times$ is called connected if each two elements $\alpha, \beta \in X$ are connected in $X$ in the sense that there is a chain $\alpha_1, \ldots, \alpha_n \in X$ with $\alpha_1 = \alpha$, $\alpha_n = \beta$ and $(\alpha_i, \alpha_{i+1}) \neq 0$, $i = 1, \ldots, n - 1$. An extended affine root supersystem $R$ is called irreducible if $R_{re} \neq \{0\}$ and $R^\times$ is connected (equivalently, $R^\times$ cannot be written as a disjoint union of two nonempty orthogonal subsets). An extended affine root supersystem $(A, (\cdot, \cdot), R)$ is called a locally finite root supersystem if the form $(\cdot, \cdot)$ is nondegenerate. A locally finite root supersystem $R$ is called a locally finite root system.
if \( R_{ns} = \{0\} \); see [3]. A subset \( S \) of a locally finite root supersystem \((A, (\cdot, \cdot), R)\) is called a sub-supersystem if the restriction of the form to \( \langle S \rangle \) is nondegenerate, \( 0 \in S \), for \( \alpha \in S \cap R^\times_{re}, \beta \in S \) and \( \gamma \in S \cap R_{ns} \) with \((\beta, \gamma) \neq 0\), \( r_\alpha(\beta) \in S \) and \( \{\gamma - \beta, \gamma + \beta\} \cap S \neq \emptyset \).

Irreducible locally finite root systems are classified in [3] and irreducible locally finite root supersystems which are not locally finite root systems are classified in [8]. According to their classifications, we have types \( \dot{A}(I, J) \), \( \dot{C}(0, J) \), \( A(\ell, \ell) \), \( B(I, J) \), \( BC(I, J) \), \( C(I, J) \), \( D(I, J) \), \( AB(1, 3) \), \( G(1, 2) \) and \( D(2, 1, \lambda) \) for irreducible locally finite root supersystems which are not locally finite root systems.

**Definition 1.3.** Suppose that \((A, (\cdot, \cdot), R)\) is a locally finite root supersystem and \( \Lambda \) is an additive abelian group. A Lie superalgebra \( L = L_0 \oplus L_1 \) is called an \((R, \Lambda)\)-graded Lie superalgebra if

- the Lie superalgebra \( L \) is equipped with a \( \langle R \rangle \)-grading \( L = \bigoplus_{\alpha \in \langle R \rangle} L^\alpha \), that is
  - \( L_0 \) as well as \( L_1 \) are \( \langle R \rangle \)-graded subspaces,
  - \( [L^\alpha, L^\beta] \subseteq L^{\alpha + \beta} \) for all \( \alpha, \beta \in \langle R \rangle \),
- the support of \( L \) with respect to the \( \langle R \rangle \)-grading is a subset of \( R \),
- \( L^0 = \sum_{\alpha \in R \setminus \{0\}} [L^\alpha, L^{-\alpha}] \),
- the Lie superalgebra \( L \) is equipped with a \( \Lambda \)-grading \( L = \bigoplus_{\lambda \in \Lambda} \lambda L \) which is compatible with the \( \langle R \rangle \)-grading on \( L \), that is
  - \( L_0 \) as well as \( L_1 \) are \( \Lambda \)-graded subspaces,
  - \( \lambda L^\alpha \) is a \( \Lambda \)-graded subspace for each \( \alpha \in R \),
  - \( [\lambda L, \mu L] \subseteq \lambda + \mu L \) for all \( \lambda, \mu \in \Lambda \),
- there is a subsystem \( \Phi \) of \( R \) such that \( Q \otimes_{Z} \text{Span}_Z R = \text{span}_Q (1 \otimes \Phi) \) and that for \( 0 \neq \alpha \in \Phi \), there are \( 0 \neq e \in 0L \cap L^\alpha \) and \( 0 \neq f \in 0L \cap L^{-\alpha} \) with \( k_\alpha := [e, f] \in L_0 \setminus \{0\} \) and for \( \beta \in R \) and \( x \in L^\beta \), \( [k_\alpha, x] = (\beta, \alpha)x \) (we call \( \{k_\alpha \mid \alpha \in \Phi \setminus \{0\} \} \) a set of toral elements and refer to \( \Phi \) as a grading subsystem).
Theorem 1.4 ([10, Thm. 2.7]). Suppose that $(\hat{A}, (\cdot, \cdot), \hat{R})$ is a locally finite root supersystem and $\Lambda$ is a torsion free additive abelian group. Suppose that $\mathcal{G} = \oplus_{\lambda \in \Lambda} \lambda \mathcal{G} = \bigoplus_{\hat{\alpha} \in \hat{R}} \mathcal{G}^{\hat{\alpha}}$ is a centerless $(\hat{R}, \Lambda)$-graded Lie superalgebra, with a grading subsystem $\Phi$, equipped with an invariant nondegenerate even supersymmetric bilinear form $(\cdot, \cdot)$. Suppose that

- for $\lambda, \mu \in \Lambda$ with $\lambda + \mu \neq 0$, $(\lambda \mathcal{G}, \mu \mathcal{G}) = \{0\}$,
- the form is nondegenerate on the span of a set of toral elements of $\mathcal{G}$,
- for $\hat{\alpha} \in \hat{R} \setminus \{0\}$ and $\lambda \in \Lambda$ with $\lambda \mathcal{G}^{\hat{\alpha}} := \mathcal{G}^{\hat{\alpha}} \cap \lambda \mathcal{G} \cap \mathcal{G}^{\hat{\alpha}} \neq \{0\}$ ($i = 0, 1$), there are $e \in \lambda \mathcal{G}^{\hat{\alpha}}$ and $f \in -\lambda \mathcal{G}^{-\hat{\alpha}}$ such that $k := [e, f] \in \mathcal{G}_{\hat{0}} \setminus \{0\}$ and for $\beta \in \hat{R}$ and $x \in \mathcal{G}^{\beta}$, $[k, x] = (\beta, \hat{\alpha})x$,
- for $\lambda \in \Lambda \setminus \{0\}$ and $i \in \{0, 1\}$ with $\lambda \mathcal{G}^{0}_i := \mathcal{G}^{\hat{i}} \cap \lambda \mathcal{G} \cap \mathcal{G}^{0} \neq \{0\}$, there are $e \in \lambda \mathcal{G}^{0}_i$ and $f \in -\lambda \mathcal{G}^{-0}_i$ such that $[e, f] = 0$ and $(e, f) \neq 0$,

then $\mathcal{G}$ is isomorphic to the core of an extended affine Lie superalgebra modulo the center.

Suppose that $\mathcal{V} = \mathcal{V}_0 \oplus \mathcal{V}_1$ is a superspace and fix bases $\{v_i \mid i \in I\}$ and $\{v_j \mid j \in J\}$ for $\mathcal{V}_0$ and $\mathcal{V}_1$ respectively, in which $I$ and $J$ are two disjoint nonempty index sets. Consider the Lie superalgebra $\mathfrak{gl}(\mathcal{V}) := \text{End}(\mathcal{V})$ under the supercommutator product and for $j, k \in I \cup J$, define

$$e_{j,k} : \mathcal{V} \rightarrow \mathcal{V}; \quad v_i \mapsto \delta_{k,i}v_j \quad (i \in I \cup J),$$

then $\mathfrak{gl}(I \cup J) := \text{span}_{\mathbb{Z}}\{e_{j,k} \mid j, k \in I \cup J\}$ is a Lie subsuperalgebra of $\mathfrak{gl}(\mathcal{V})$. For $A = \sum_{i,j \in I \cup J} a_{i,j}e_{i,j} \in \mathfrak{gl}(I \cup J)$, define the supertrace of $A$ to be

$$\text{str}(A) := \sum_{i \in I} a_{i,i} - \sum_{j \in J} a_{j,j}.$$

Example 1.5. Suppose that $I$ and $J$ are two infinite index sets and $\bar{I}$ and $\bar{J}$ are copies of $I$ and $J$ respectively such that $I, J, \bar{I}$ and $\bar{J}$ are mutually disjoint. For $i \in I$ (resp. $j \in J$), we denote by $\bar{i}$ (resp. $\bar{j}$) the element of $\bar{I}$ (resp. $\bar{J}$) corresponding to $i$ (resp. $j$) with respect to a fixed identification. Suppose that $\mathcal{V}_0$ is a vector space with a basis $\{v_i, v_i \mid i \in I\}$ and $\mathcal{V}_1$ is a vector space with a basis $\{v_j, v_j \mid j \in J\}$. Define the form

$$(\cdot, \cdot)_0 : \mathcal{V}_0 \times \mathcal{V}_0 \rightarrow \mathbb{F}$$

$$(v_i, v_r) \mapsto 0, \quad (v_j, v_r) \mapsto 0, \quad (v_i, v_r) \mapsto \delta_{i,r}, \quad (v_i, v_r) \mapsto \delta_{i,r} \quad (i, r \in I),$$
and the form
\[(\cdot, \cdot)_1 : \mathcal{V}_1 \times \mathcal{V}_1 \to \mathbb{F} \]
\[\quad (v_j, v_s) \mapsto 0, \quad (v_j, v_s) \mapsto -\delta_{j,s}, \quad (v_j, v_s) \mapsto \delta_{j,s} \quad (j, s \in J).\]

The form \((\cdot, \cdot) := (\cdot, \cdot)_0 \oplus (\cdot, \cdot)_1\) is a nondegenerate supersymmetric even bilinear form on the superspace \(\mathcal{V} := \mathcal{V}_0 \oplus \mathcal{V}_1\). We next consider the Lie subsuperalgebra
\[\mathfrak{g} := o(\mathfrak{I}, \mathfrak{J}) := \{X \in \mathfrak{gl}(\mathfrak{I} \oplus \mathfrak{\bar{I}} \oplus \mathfrak{J} \oplus \mathfrak{\bar{J}}) \mid (Xv, w) = \cdots \} \quad (v, w \in \mathcal{V}).\]

For \(i \in \mathfrak{I}\) and \(j \in \mathfrak{J}\), take
\[h_i := e_{i,i} - e_{i,i} \quad \text{and} \quad d_j := e_{j,j} - e_{j,j},\]
and set
\[\mathcal{H} := \text{span}_{\mathbb{F}}\{h_i, d_j \mid i \in \mathfrak{I}, j \in \mathfrak{J}\}.\]

Define
\[(\cdot, \cdot) : \mathcal{G} \times \mathcal{G} \to \mathbb{F}; \quad (x_1, x_2) \mapsto \text{str}(x_1x_2) \quad (x_1, x_2 \in \mathcal{G}).\]

Also for \(r \in \mathfrak{I}\) and \(s \in \mathfrak{J}\), define
\[\epsilon_r : \mathcal{H} \to \mathbb{F}; \quad h_i \mapsto \delta_{r,i}, \quad d_j \mapsto 0 \quad (i \in \mathfrak{I}, j \in \mathfrak{J})\]
\[\delta_s : \mathcal{H} \to \mathbb{F}; \quad h_i \mapsto 0, \quad d_j \mapsto \delta_{j,s} \quad (i \in \mathfrak{I}, j \in \mathfrak{J}).\]

We know that \((\mathcal{G}, \mathcal{H}, (\cdot, \cdot))\) is an extended affine Lie superalgebra with \(\mathcal{G}^0 = \mathcal{H}\) and root system
\[R = \{\pm\epsilon_i \pm \epsilon_r, \pm\delta_j \pm \delta_s, \pm\epsilon_i \pm \delta_j \mid j, s \in \mathfrak{J}, i, r \in \mathfrak{I}, i \neq r\}.\]

which is an irreducible locally finite root supersystem of type \(D(\mathfrak{I}, \mathfrak{J})\). Using the same notation as in the paper, for \(i \in \mathfrak{I}\) and \(j \in \mathfrak{J}\), we have
\[t_{\epsilon_i} = \frac{1}{2}(\epsilon_{i,i} - \epsilon_{i,i}) \quad \text{and} \quad t_{\delta_j} = -\frac{1}{2}(\epsilon_{j,j} - \epsilon_{j,j}).\]

Moreover, for \(\theta, \theta' \in \{\pm\epsilon_i, \pm\delta_j\}\) with \(\theta + \theta' \in R \setminus \{0\}\), we have \(\mathcal{G}^{\theta + \theta'} = \mathbb{F}x_{\theta,\theta'}\) in which \(x_{\theta,\theta'}\) is given as in the following table:
for $i, r, t \in I$ and $j, s, k \in J$ with $i \neq r$ and $j, k \in J$. We also know that $\mathcal{V} = \mathcal{V}_0 \oplus \mathcal{V}_1$, under the natural action $xv := x(v)$, for all $x \in \mathcal{G}$ and $v \in \mathcal{V}$, is a $\mathcal{G}$-module. The $\mathcal{G}$-module $\mathcal{V}$ has a weight space decomposition $\mathcal{V} = \bigoplus_{\alpha \in \{\pm \epsilon, \pm \delta\}} \mathcal{V}^\alpha$ with respect to $\mathcal{H}$, where

\[ \mathcal{V}^{\epsilon_i} = F v_i, \quad \mathcal{V}^{-\epsilon_i} = F v_i, \quad \mathcal{V}^{\delta_j} = F v_j, \quad \mathcal{V}^{-\delta_j} = F v_j; \quad (i \in I, j \in J). \]

For $u, v \in \mathcal{V}$, define $\gamma_{u, v} : \mathcal{V} \rightarrow \mathcal{V}$ mapping $w \in \mathcal{V}$ to $(v, w)u - (-1)^{|u||w|}(u, w)v$. An easy verification shows that for $u, v \in \mathcal{V}_0 \cup \mathcal{V}_1$, $\gamma_{u, v}$ is an element of $\mathcal{G}_{|u|+|v|}$ and that $\gamma_{u, v} = -(-1)^{|u||v|}\gamma_{v, u}$. We also have the following table:

\[
\begin{array}{cccc}
(u, v) & \gamma_{u, v}(v_i) & \gamma_{u, v}(v_j) & \gamma_{u, v}(v_k) \\
(v_i, v_r) & 0 & \delta_{r, t}v_i - \delta_{i, t}v_r & 0 & 0 \\
(v_i, v_r) & -\delta_{i, t}v_r & \delta_{r, t}v_i & 0 & 0 \\
(v_j, v_r) & \delta_{r, t}v_i - \delta_{i, t}v_r & 0 & 0 & 0 \\
(v_j, v_r) & 0 & 0 & 0 & \delta_{s, k}v_j + \delta_{j, k}v_s \\
(v_j, v_r) & 0 & 0 & -\delta_{j, k}v_i & \delta_{s, k}v_j \\
(v_j, v_r) & 0 & 0 & -\delta_{s, k}v_j - \delta_{j, k}v_s & 0 \\
(v_j, v_r) & -\delta_{i, t}v_j & 0 & 0 & \delta_{j, k}v_i \\
(v_j, v_r) & -\delta_{i, t}v_j & 0 & 0 & \delta_{j, k}v_i \\
(v_j, v_r) & 0 & -\delta_{i, t}v_j & -\delta_{j, k}v_i & 0 \\
(v_j, v_r) & -\delta_{i, t}v_j & 0 & -\delta_{j, k}v_i & 0 \\
\end{array}
\]

for $i, r, t \in I$ and $j, s, k \in J$. In particular,

\[ \gamma_{v_i, v_i} = -2t_{\epsilon_i} \quad \text{and} \quad \gamma_{v_j, v_j} = 2t_{\delta_j}; \quad i \in I, j \in J. \quad (2) \]
Set
\[ \mathcal{L} := (G \otimes F[t^2, t^{-2}]) \oplus (V \otimes F[t^2, t^{-2}]). \]

**Proposition 1.6.** \( \mathcal{L} \) together with the bilinear extension of the following brackets

\[
\begin{align*}
-(1)^{|x||y|}[y \otimes t^k, x \otimes t^k] & = [x \otimes t^k, y \otimes t^k] := [x, y] \otimes t^{k+\ell}, \\
-(1)^{|x||u|}[u \otimes t^m, x \otimes t^k] & = [x \otimes t^k, u \otimes t^m] := xu \otimes t^{k+m}, \\
-(1)^{|u||v|}[v \otimes t^m, u \otimes t^m] & = [u \otimes t^m, v \otimes t^m] := \gamma_{u,v} \otimes t^{m+n},
\end{align*}
\]

for all \( x, y \in G, u, v \in V, k, \ell \in 2\mathbb{Z}, m, n \in 2\mathbb{Z} + 1 \) is a Lie superalgebra with

\[
\mathcal{L}_0 = (G_0 \otimes F[t^2, t^{-2}]) \oplus (V_0 \otimes F[t^2, t^{-2}])
\]

and

\[
\mathcal{L}_1 = (G_1 \otimes F[t^2, t^{-2}]) \oplus (V_1 \otimes F[t^2, t^{-2}]).
\]

**Proof.** For \( u, v, w \in V, x, y, z \in G \) and \( \ell, m, n \in \mathbb{Z} \), we have

\[
[\gamma_{u,v}, x](w) = \gamma_{u,v}xw - (1)^{|x||([u]+[v])|}x\gamma_{u,v}w
\]

\[=
(v, xw)u - (-1)^{|u||v|}(u, xw)v
\]

\[= -(1)^{|x||([u]+[v])|}((v, w)xu - (-1)^{|u||v|}(u, w) xv)
\]

\[= -(1)^{|v||x|}(xv, w)u + (-1)^{|u||v|}(1)^{|u||z|}(xu, w)v
\]

\[= -(1)^{|u||z|}(xv, w)u + (-1)^{|u||v|}(1)^{|u||z|}(xu, w)v
\]

\[= -(1)^{|u||z|}(xv, w)u + (-1)^{|u||v|}(1)^{|u||z|}(xu, w)v
\]

\[= -(1)^{|u||z|}(xv, w)u + (-1)^{|u||v|}(1)^{|u||z|}(xu, w)v
\]

\[= -(1)^{|u||z|}(xv, w)u + (-1)^{|u||v|}(1)^{|u||z|}(xu, w)v
\]

\[= -(1)^{|u||z|}(xv, w)u + (-1)^{|u||v|}(1)^{|u||z|}(xu, w)v
\]

\[= -(1)^{|u||z|}(xv, w)u + (-1)^{|u||v|}(1)^{|u||z|}(xu, w)v
\]

So

\[
[\gamma_{u,v}, x] = -(1)^{|v||x|}\gamma_{u,v} + (-1)^{|u||v|}(1)^{|u||z|}\gamma_{v,x,u}.
\]
Now we have the following equalities:

\[
[u \otimes t^m, v \otimes t^n], x \otimes t^\ell] = [\gamma_{u,v} \otimes t^{m+n}, x \otimes t^\ell]
\]

\[
= [\gamma_{u,v}, x] \otimes t^{m+n+\ell}
\]

\[
= -(-1)^{|u||v|}[\gamma_{u,v} \otimes t^{m+n+\ell}
\]

\[
+(-1)^{|u||v|}(-1)^{|u||x|}[\gamma_{v,x} \otimes t^{m+n+\ell}] = [u \otimes t^m, [v \otimes t^n, x \otimes t^\ell]]
\]

\[
= -(-1)^{|u||v|}[u \otimes t^m, [u \otimes t^m, x \otimes t^\ell]]
\]

and

\[
[u \otimes t^m, v \otimes t^n], w \otimes t^\ell] = [\gamma_{u,v} \otimes t^{m+n}, w \otimes t^\ell]
\]

\[
= [\gamma_{u,v}, w] \otimes t^{m+n+\ell}
\]

\[
= -(-1)^{|u||v|}(-1)^{|w||v|}[\gamma_{w,u} \otimes t^{m+n+\ell}
\]

\[
+(-1)^{|u||v|}(-1)^{|w||u|}[\gamma_{v,w} \otimes t^{m+n+\ell}] = [u \otimes t^m, [v \otimes t^n, w \otimes t^\ell]]
\]

\[
+[u \otimes t^m, [u \otimes t^m, w \otimes t^\ell]]
\]

Also we have

\[
[x \otimes t^m, y \otimes t^n], u \otimes t^\ell] = [[x, y] \otimes t^{m+n}, u \otimes t^\ell]
\]

\[
= [x, y]u \otimes t^{m+n+\ell}
\]

\[
=(x(yu) - (-1)^{|x||y|}y(xu)) \otimes t^{m+n+\ell}
\]

\[
=[x \otimes t^m, yu \otimes t^{n+\ell} - (-1)^{|x||y|}y \otimes t^n, xu \otimes t^{m+\ell}] = [x \otimes t^m, [y \otimes t^n, u \otimes t^\ell]] - (-1)^{|x||y|}[y \otimes t^n, [x \otimes t^m, u \otimes t^\ell]]
\]

and

\[
[x \otimes t^m, y \otimes t^n], z \otimes t^\ell] = [[x, y] \otimes t^{m+n}, z \otimes t^\ell]
\]

\[
=[[x, y], z] \otimes t^{m+n+\ell}
\]

\[
=[[x, [y, z]]] \otimes t^{m+n+\ell} - (-1)^{|x||y|}[y, [x, z]] \otimes t^{m+n+\ell}
\]

\[
=[x \otimes t^m, [y \otimes t^n, z \otimes t^\ell]] - (-1)^{|x||y|}[y \otimes t^n, [x \otimes t^m, z \otimes t^\ell]]
\]
This completes the proof.

\[\] \[
\]

Proposition 1.7. Define the bilinear form \((\cdot, \cdot) : \mathcal{L} \times \mathcal{L} \rightarrow \mathbb{F}\) by
\[
(x \otimes t^k, y \otimes t^\ell) := \delta_{k,-\ell} \text{str}(xy),
\]
\[
(x \otimes t^k, u \otimes t^m) := 0,
\]
\[
(u \otimes t^m, v \otimes t^n) := 2\delta_{m,-n}(u, v).
\]

Then it is a nondegenerate invariant even supersymmetric bilinear form.

Proof. It is immediate that the form is nondegenerate, even and supersymmetric. So we just need to show that it is invariant. It is sufficient to show that \(((u \otimes t^m, v \otimes t^n), x \otimes t^m) = (u \otimes t^n, [v \otimes t^\ell, x \otimes t^m])\) for \(u, v \in \mathcal{V}, x \in \mathcal{G}, \ell, n \in 1 + 2\mathbb{Z}\) and \(m \in 2\mathbb{Z}\). Suppose that \(t \in I \cup \bar{I} \cup \mathcal{J} \cup \bar{J}\) and that \(u, v\) are homogeneous. If \(u = \sum_{k \in I \cup \bar{I} \cup \mathcal{J}} \delta_{[i,j]} r_k v_k\) and \(v = \sum_{k \in I \cup \bar{I} \cup \mathcal{J}} \delta_{[i,j]} s_k v_k\), since \(u, v\) are homogenous, \(u = \sum_{k \in I \cup \bar{I} \cup \mathcal{J}} \delta_{[i,j]} r_k v_k\) and \(v = \sum_{k \in I \cup \bar{I} \cup \mathcal{J}} \delta_{[i,j]} s_k v_k\). So
\[
\gamma_{u,v,x} = (u, xv) = \sum_k (-1)^{|u||v|} (u, x v) \delta_{[i,j]} (u, x v) 
\]
Theorem 1.8. \( \mathcal{L} \) is the centerless core of an extended affine Lie superalgebra.

Proof. Set

\[
\mathfrak{R} := R \cup \{ \pm \epsilon_i, \pm \delta_j \mid i \in I, j \in J \} \\
= \{ \pm \epsilon_i, \pm \delta_j, \pm \epsilon_r, \pm \delta_s, \pm \epsilon_i \pm \delta_j, \pm \epsilon_r \pm \delta_s, \pm \epsilon_i \pm \delta_j \pm \delta_s \mid j, s \in J, i, r \in I, i \neq r \}
\]

and \( A := \text{span}_\mathbb{F} \mathfrak{R} \). Define

\[
(\cdot, \cdot) : A \times A \rightarrow \mathbb{F} \\
(\epsilon_i, \epsilon_r) := \frac{1}{2} \delta_{i,r}, \quad (\delta_j, \delta_s) := -\frac{1}{2} \delta_{j,s}, \quad (\epsilon_i, \delta_j) := 0 \quad (i, r \in I, j, s \in J).
\]

Then \( \mathfrak{R} \) is an irreducible locally finite root supersystem of type \( B(I, J) \).

For \( \alpha \in \mathfrak{R} \), define

\[
\mathcal{L}^\alpha := \begin{cases} 
G^\alpha \otimes \mathbb{F} [t^2, t^{-2}] & \text{if } \alpha \in R \setminus \{0\} \\
V^\alpha \otimes \mathbb{F} [t^2, t^{-2}] & \text{if } \alpha \in \{ \pm \epsilon_i, \pm \delta_j \mid i \in I, j \in J \} \\
\mathcal{H} \otimes \mathbb{F} [t^2, t^{-2}] & \text{if } \alpha = 0.
\end{cases}
\]

Also for \( m \in \mathbb{Z} \), define

\[
m_\mathcal{L} := \begin{cases} 
G \otimes \mathbb{F} t^m & \text{if } m \in 2\mathbb{Z} \\
V \otimes \mathbb{F} t^m & \text{if } m \in 1 + 2\mathbb{Z}.
\end{cases}
\]

Consider Lemma 1.1 and (2) and suppose \( \alpha \in R \setminus \{0\} \) and \( \beta \in \mathfrak{R} \setminus (R \cup \{0\}) \). Fix \( e_\alpha \in G^\alpha \), \( f_\alpha \in G^{-\alpha} \), \( u_\beta \in V^\beta \) and \( v_\beta \in V^{-\beta} \) with \([e_\alpha, f_\alpha] = t_\alpha \) and \( \gamma_{u_\beta, v_\beta} = t_\beta \). Then for

\[
x^m_\gamma := \begin{cases} 
electron m t^m & \text{if } \gamma \in R \setminus \{0\}, m \in 2\mathbb{Z} \\
u_\gamma t^m & \text{if } \gamma \in \mathfrak{R} \setminus (R \cup \{0\}), m \in 1 + 2\mathbb{Z}
\end{cases}
\]

and

\[
y^m_\gamma := \begin{cases} 
f_\gamma t^m & \text{if } \gamma \in R \setminus \{0\}, m \in 2\mathbb{Z} \\
v_\gamma t^m & \text{if } \gamma \in \mathfrak{R} \setminus (R \cup \{0\}), m \in 1 + 2\mathbb{Z},
\end{cases}
\]

we have \( h^m_\gamma := [x^m_\gamma, y^m_\gamma] = t_\gamma \otimes 1 \) and for each \( \eta \in \mathfrak{R} \) and \( x \in \mathcal{L}^\eta \), we have

\[
[h^m_\gamma, x] = (\eta, \gamma) x.
\]

Now \( \mathcal{L} \) is an \( (\mathfrak{R}, \mathbb{Z}) \)-graded Lie superalgebra. Moreover, using Propositions 1.4 and 1.7, one gets that \( \mathcal{L} \) is the centerless core of an extended affine Lie superalgebra.

\qed
References


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(Received February 1, 2015)